On Numerical Methods for Real Solutions of One Variable Nonlinear Equations

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Outline

Background and Our three important questions

Equation is perhaps the most elementary notion in the field of mathematics. A formal statement of equality of a mathematical or logical expression is defined as an equation.
Researchers and philosophers have delved into the art of solving equations for more than two millennia. But...

- **WHEN** did it all formally started?
- **HOW** did our ancestors solve them (and how do we solve them)?
  To answer the second question we need to go to section 2
- And, very important question for a researcher and non-mathematician alike, **WHY** do we need equations and their solutions?
  Section 6 is shines light on this question.

When did it all started? I

The very first formal conception of what we know as an equation can be found into work of a 16th century French algebraist François Viète.
But is that really when we began? Short answer: No.
Long answer: Equations have been known to humans long before Viète.

- Babylonian, Ancient {Indian, Egyptian and Greek} civilizations were well aware of the concept of equality and equations.
- Babylonians actively solved quadratic and cubic equations. And factored the terms to solve these equations.
- Egyptians, however, used one of the numerical techniques (regula-falsi) we know today to solve equations.

When did it all started? II

- Indians, were extremely active (and probably far advanced than their counterparts), in understanding and solving equations.

  Brahmagupta, for example, gave solution to generalised quadratic equations which we use even today.

  Whereas, Bhaskara II gave solution to what we today know as Pell's equation ($x^2 - ny^2 = 1$).

Analytical Methods

This is what solving equations meant for our ancestors

For equation: $ax^2 + bx + c = 0$ the solution is:

$\left(x = \frac{-b \pm \sqrt{b^2 - 4*a*c}}{2a}\right)$ (Brahmagupta)

To solve the equation: $y^3 + Ay = B$ we first substitute:

$A = 3st$ and $B = s^3 - t^3$ and then we solve for:

$y = s - t$ (Cardano)

Numerical Methods

This is how we solve equations

For equation: $f(y) = ye^{y^2} - sin^2(y) + 3cos(y) + 5 = 0$ the solution can be obtained using:

Taking initial approximation $y_0 = -1.9$ and then applying Newton's method until the difference between two consecutive iterates y_{t+1} and y_t is not less than the assume tolerance $\epsilon = 10^{-d}$

Where Newton's method is:

$$y_{t+1} = y_t - \frac{f(y_t)}{f'(y_t)}$$

Literature survey I

We have divided survey of over 100 research articles in three different sections. And from our survey we have observed a particular pattern in all the methods in the gamut of available literature. The sections are as follows:

- First section consists the development before the 20^{th} century.
  The major development include Newton-Raphson Method and Halley's Method [16], proof of quadratic convergence of Newton's method by Fourier [13] and the first semilocal convergence theorem by Cauchy [10].
- Second section focuses on the development of numerical methods as well as developments of crucial concept used to introduce novel methods.
  The major progress during the century include concept of Banach Space [5], Multiplicative Calculus [15], analytical study of Halley's method [9, 2], and development of novel numerical methods [17, 21, 1].

Literature survey II

- Last section covers the literature of the 21^{st} century and is divided in seven subsections due the nature and motive of the different kinds of studies. These subsections are divided as follows:
  1. Development of Novel concepts [6, 30]
  2. Methods based on Newton-Raphson Methods [18, 14, 25]
  3. Multi-step methods [22, 12, 3]
  4. Methods based Halley, Steffensen and others [20, 19]
  5. Derivative free iterative schemes [32, 7]
  6. Methods using decomposition technique and multiplicative calculus [4, 31]
  7. Fixed Point Iteration based methods [8, 24]

Based on the detailed survey of different methods developed from 1694 [16] till 2020 [3], we deduced that all methods to solve one variable equations can be categorised in 4 Major Families.

Four Major Families

The four major families are:
(1) Methods developed for (and from) the equation $f(x) = 0$ and using Newtonian Calculus (Newton based methods)
(2) Methods developed for (and from) the equation $\phi(x) = x$ and using Newtonian Calculus (Fixed point based methods)
(3) Methods developed for (and from) the equation $G(x) = 1$ and using multiplicative ($*$) derivative (Multiplicative based methods)
(4) Methods developed for (and from) the equation $G(x) = 1$ and using Volterra (π) derivative (Volterra based methods)
In the brackets are the aliases given by us to the families for the ease of understanding. And these aliases represent the central idea on which the family is based [1].

[1] Newton based method is meant as the methods based on Newtonian Calculus and not as the methods derived from the Newton-Raphson method

Definition

[29]

A nonempty set Y with a map $d : Y \times Y \to \mathbb{R}$ is called a metric space if the map d has the following properties:

(1) $d(y, x) \geq 0 \quad y, x \in Y$

(2) $d(y, x) = 0$ if and only if $y = x$

(3) $d(y, x) = d(x, y) \quad y, x \in Y$

(4) $d(y, x) \leq d(y, z) + d(z, x) \quad y, x, z \in Y$ (Triangular Inequality)

The map d is called the metric on Y or sometimes the distance function on Y. The phrase "(Y, d) is a metric space" means that d is a distance function on the set Y.

Definition

Let d be a metric on a set Y. A sequence $\{y_t\}_{t\geq 1}$ in the set Y is said to be a Cauchy sequence if, for every $\varepsilon > 0$, there exists a natural number t_0 such that $d(y_t, y_m) < \varepsilon$ whenever $t \geq t_0$ and $m \geq t_0$.

Definition

A metric space (Y, d) is called **complete**, if every Cauchy sequence converges in it.

Definition

Let (Y, d) be a metric space. A mapping T of Y into itself is said to be a contraction mapping (or contraction) if there exists a real number $\gamma, 0 < \gamma < 1$, such that

$$d(Ty, Tx) \leq \gamma * d(y, x)$$

for all $y, x \in Y$

Preliminaries III

Definition

A point $y \in Y$ is called a **Fixed Point** of the mapping $T : Y \to Y$ if $Ty = y$.

Theorem

(Contraction Mapping Theorem) Let $T : Y \to Y$ *be a contraction of the complete metric space* (Y, d) *then* T *has a unique fixed point.*

Definition

[6]

Let $B \subset \mathbb{R}$ be an open interval and let $G : B \to \mathbb{R}^+$ be continuous then the $*$ derivative or multiplicative derivative of G is defined as:

$$G^*(y) = \left(\frac{G(y+h)}{G(y)} \right)^{\frac{1}{h}} \tag{1}$$

Preliminaries IV

Therefore,

$$\begin{aligned} G^*(y) &= \left(\frac{G(y+h)}{G(y)}\right)^{\frac{1}{h}} \\ &= \left(1 + \frac{G(y+h) - G(y)}{G(y)}\right)^{\frac{G(y)}{G(y+h)-G(y)} \cdot \frac{G(y+h)-G(y)}{h} \cdot \frac{1}{G(y)}} \\ &= e^{\frac{G'(y)}{G(y)}} = e^{(\ln \circ G)'(y)} \end{aligned}$$

where $(\ln \circ G)(y) = \ln G(y)$.

Preliminaries V

Definition

[23]
Let $B \subset \mathbb{R}$ be an open interval and let $G : B \to \mathbb{R}^+$ be continuous then the π derivative or Volterra derivative of G is defined as:

$$G^{\pi}(y) = \left(\frac{G((1+h)y)}{G(y)}\right)^{\frac{1}{h}}$$

$$\implies G^{\pi}(y) = G^{*}(y)^{y}$$

$$G^{\pi\pi}(y) = G^{**}(y)^{y^2} \cdot G^{*}(y)^{y}$$

$$G^{\pi\pi\pi}(y) = G^{***}(y)^{y^3} \cdot G^{**}(y)^{3y^2} \cdot G^{*}(y)^{y}$$

Definition

Suppose that $\alpha = \lim_{t\to\infty} y_{t+1}$. We say that the sequence $\{y_t\}$ converges to α **with at least order** $p > 1$ if there exist a sequence (ϵ_t) of positive real numbers converging to 0, and $\mu > 0$, such that

$$|y_t - \alpha| \leq \epsilon_t, \quad t = 0, 1, 2, .., \quad \textit{and} \quad \lim_{t\to\infty} \frac{\epsilon_{t+1}}{\epsilon_t^p} = \mu. \qquad (2)$$

Novel Methods

The novel methods are:

$$y_{t+1} = y_t - \frac{2(y_t - \phi(y_t))(1 - \phi'(y_t))}{2[1 - \phi'(y_t)]^2 + (y_t - \phi(y_t))\phi''(y_t)} \tag{3}$$

$$y_{t+1} = y_t + \frac{(1 - \phi'(y_t)) - \sqrt{[1 - \phi'(y_t)]^2 + 2(y_t - \phi(y_t))\phi''(y_t)}}{\phi''(y_t)} \tag{4}$$

$$y_{t+1} = y_t + \frac{2(y_t - \phi(y_t))}{(1 - \phi'(y_t)) + \sqrt{[1 - \phi'(y_t)]^2 + 2(y_t - \phi(y_t))\phi''(y_t)}} \tag{5}$$

Convergence Analysis and Efficiency Index

- All three of are Novel methods are cubic convergent and have efficiency index of 1.4422 which is greater than Newton's method.

First Application I

Example

[11] In lumped system analysis, to study the transient heat conduction in one dimension, the transcendental equation of the form $y \tan(y) = Bi$ is used. Which is Eigenfunction of the exact solution of one-Dimensional transient conduction problem and y is the eigenvalue.

The solution of transient heat transfer in one dimension can be given as:

$$y \cdot \tan(y) - 1 = 0 \tag{6}$$

on assuming the value of $Bi = 1$. For equation **(6)** we have $\alpha = 0.8603335890193797 6...$ and we consider $y_0 = 1.47$. And, The corresponding fixed point function and multiplicative functions are: $\phi(y) = \cot^{-1}(y)$ and $G(y) = y \cdot \tan(y)$.

First Application II

Table: Numerical Comparison for all nine methods for example 1

Method	NOI	$\lvert y_{t+1} - y_t \rvert$	$\lvert f(y_{t+1}) \rvert$	OC	EI
FH	4	$5.04 \cdot 10^{-27}$	$6.72 \cdot 10^{-81}$	3	1.4422
M/V NR	6	$6.56 \cdot 10^{-19}$	$1.59 \cdot 10^{-37}$	2	1.2599
Halley	4	$1.05 \cdot 10^{-16}$	$9.26 \cdot 10^{-49}$	3	1.4422
DMS	4	$7.24 \cdot 10^{-24}$	$5.95 \cdot 10^{-71}$	3	1.4422
NR	9	$6.75 \cdot 10^{-18}$	$2.14 \cdot 10^{-34}$	2	1.4142
VH	*Fails*	–	–	–	–
SS	4	$7.13 \cdot 10^{-24}$	$5.49 \cdot 10^{-71}$	3	1.4422
FNR	5	$4.15 \cdot 10^{-16}$	$9.93 \cdot 10^{-32}$	2	1.4142
MH	*Fails*	–	–	–	–

Second Application I

Example

[26] Chemical reactor problem:
In regard to fraction transformation in a chemical reactor, we consider

$$\frac{y}{1-y} - 5\ln\left[\frac{0.4(1-y)}{0.4-0.5y}\right] + 4.45977 = 0 \tag{7}$$

where the variable y denotes a fractional transformation of a particular species Z in the chemical reactor problem. Also, we have $\alpha = 0.757396246253753879...$ and $y_0 = 0.86$, respectively. And, The corresponding fixed point function and multiplicative functions are: $\phi(y) = \frac{4}{5}\left(1 - \frac{1-x}{\exp\left(\frac{\frac{x}{1-x}+\frac{445977}{100000}}{5}\right)}\right)$ and $G(y) = \frac{100000}{445977}\left(5\ln\left[\frac{0.4(1-y)}{0.4-0.5y}\right] - \frac{y}{1-y}\right)$.

Second Application II

Table: Numerical Comparison for all nine methods for example 2

Method	NOI	$\lvert y_{t+1} - y_t \rvert$	$\lvert f(y_{t+1}) \rvert$	OC	EI
FH	4	$1.48 \cdot 10^{-24}$	$2.47 \cdot 10^{-70}$	3	1.4422
M/V NR	*Fails*	–	–	–	–
Halley	*Fails*	–	–	–	–
DMS	4	$7.64 \cdot 10^{-22}$	$3.85 \cdot 10^{-65}$	3	1.4422
NR	*Fails*	–	–	–	–
VH	*Fails*	–	–	–	–
SS	4	$7.23 \cdot 10^{-22}$	$3.35 \cdot 10^{-65}$	3	1.4422
FNR	5	$4.55 \cdot 10^{-21}$	$5.98 \cdot 10^{-40}$	2	1.4142
MH	*Fails*	–	–	–	–

Third Application I

Example

[28] Let us assume an adiabatic flame temperature equation, which is given by

$$\frac{\gamma}{3}(y^3 - 2983) + \frac{\beta}{2}(y^2 - 2982) + \alpha(y - 298) + \Delta H = 0 \qquad (8)$$

where $\gamma = 0.283 \times 10^{-6}, \beta = 2.298 \times 10^{-3}, \alpha = 7.256$ and $\Delta H = -57798$. The equation has a root $\alpha = 4300.793148061962046381220388078551\,17$. Moreover, we have assumed the initial approximation is $y_0 = 4307$ for this problem. And, The corresponding fixed point function and multiplicative functions are: $\phi(y) = \left(\frac{-\Delta H + \frac{2983\gamma}{3} + \frac{2982\beta}{2} + 298\alpha}{\frac{\gamma y^2}{3} + \frac{\beta y}{2} + \alpha} \right)$ and $G(y) = \frac{\frac{\gamma}{3}(y^3 - 2983) + \frac{\beta}{2}(y^2 - 2982) + \alpha(y - 298)}{-\Delta H}$.

Third Application II

Table: Numerical Comparison for all nine methods for example 3

Method	NOI	$\lvert y_{t+1} - y_t \rvert$	$\lvert f(y_{t+1}) \rvert$	OC	EI
FH	4	$2.28 \cdot 10^{-29}$	$3.02 \cdot 10^{-94}$	3	1.4422
M/V NR	6	$8.83 \cdot 10^{-23}$	$1.53 \cdot 10^{-47}$	2	1.2599
Halley	4	$3.21 \cdot 10^{-26}$	$5.18 \cdot 10^{-84}$	3	1.4422
DMS	4	$4.17 \cdot 10^{-28}$	$3.81 \cdot 10^{-90}$	3	1.4422
NR	6	$3.24 \cdot 10^{-22}$	$2.49 \cdot 10^{-46}$	2	1.4142
VH	4	$1.26 \cdot 10^{-26}$	$2.46 \cdot 10^{-85}$	2	1.3161
SS	4	$3.96 \cdot 10^{-28}$	$3.51 \cdot 10^{-90}$	3	1.4422
FNR	6	$1.38 \cdot 10^{-25}$	$1.47 \cdot 10^{-53}$	2	1.4142
MH	4	$1.26 \cdot 10^{-26}$	$2.46 \cdot 10^{-85}$	2	1.3161

Comparative Study I

Volterra Based Methods:

- Owing to its definition, the Volterra based methods simply reduces to multiplicative based methods. For example the Volterra Newton-Raphson method simply reduce to Multiplicative NR.

$$y_{t+1} = y_t - y_t \cdot \frac{\log G(y_t)}{\log G^{\pi}(y_t)}$$

$$y_{t+1} = y_t - \frac{\log G(y_t)}{\log G^{*}(y_t)}$$

- Moreover, from the previous chapter it can be observed that when implemented to solve equations, method of both families yield exact same results. Hence, there is no significance of this family, and further development of Volterra based method may be discontinued.

Comparative Study II

Multiplicative Methods:

- The definition of multiplicative calculus yields $\ln(G^*(y_t)) = \frac{G'(y_t)}{G(y_t)}$. Which shows the quadratic convergent NR method of this family

$$y_{t+1} = y_t - \frac{\log G(y_t)}{\log G^*(y_t)}$$

  has to evaluate three functions per iteration. Which yields efficiency of $2^{\frac{1}{3}} \approx 1.2599$.
- This displays that the family does not obtain the optimal efficiency for any method, as to achieve the convergence order p, any method from the family needs to evaluate $p+1$ functions.

Comparative Study III

Newton based Methods:

- By far the most research has been done in the domain of this family. However, they are very slow with respect to their other three counterparts.
- Table 1 and 3 for example 1 and 3 show that even when Newton-Raphson and Halley's method are convergent, they converge to the root slower with respect to their counterparts.

Fixed point based methods:

- All the fixed point based are developed from stable fixed point iteration method.
- Once the convergence criteria for the fixed point method $|\phi'(y)| < 1 \ \forall y$ is satisfied, all the methods based on the same converge.
- Table 2 illustrates that even in the extreme cases when all other three families fail to converge, fixed point based methods yield powerful results.

Conclusion

From our study, we have arrived at following conclusions

- There are four major categories under which any of the developed method can be categorised
- For one step methods, the optimal order of convergence is three with optimal efficiency index $3^{\frac{1}{3}} \approx 1.4422$
- Volterra based methods are reduced to Multiplicative methods and yield the same results as the latter
- Multiplicative methods are never efficient
- Newton based methods achieve optimal efficiency but they do not achieve better accuracy with respect to other families

Based on all of the above conclusions we can finally deduce that:

- The fixed point based family of methods achieve highest accuracy while maintaining optimal efficiency and therefore, they are comparatively more significant than all other three families and can be purposefully implemented to different real life problems

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