

ADDITIONAL APPLICATIONS OF POWER-BOUNDED DOUBLE SUMS

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ABSTRACT. We study finite and infinite sums on the integer region $1 \leq i \leq n, i \leq j \leq \lfloor i^\rho \rfloor$, with $\rho > 1$. Counting by columns gives an explicit multiplicity that supports seven applications: lattice counts and skew diagrams, colored partitions, coprime pairs, finite probability laws, two-rook placements, relative canonical products, and finite determinants with Gaussian normalizations. The quadratic case has especially simple formulas, including an exact totient identity and closed finite moments. A square-root deficit reduces the genus of a relative product and gives a quantitative tail bound. Each construction is stated with its counting conventions and proved directly; the classical combinatorial and analytic ingredients are identified locally.

1. THE REGION AND ITS COLUMN COUNTS

The level-count approach to rounded sums $[1, 2]$ extends to integer pairs by counting columns. We use one power-bounded region throughout; its multiplicities serve as weights in sums and exponents in products.

Fix $\rho > 1$ and $n \in \mathbb{N}$, and set

$$\mathcal{R}_{\rho,n} = \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq n, i \leq j \leq \lfloor i^\rho \rfloor\}, \quad M = \lfloor n^\rho \rfloor. \quad (1.1)$$

The coupled condition $j \leq i^p/j^q$ gives this region with $\rho = p/(q+1)$, provided $p > q+1 > 1$. Put $[x]_+ = \max(x, 0)$.

Proposition 1.1 (Column multiplicities). *For $1 \leq j \leq M$, the number of pairs with second coordinate j is*

$$c_{\rho,n}(j) = \lfloor \min(n, j) - \lceil j^{1/\rho} \rceil + 1 \rfloor_+. \quad (1.2)$$

For any scalar weight f ,

$$\sum_{i=1}^n \sum_{j=i}^{\lfloor i^\rho \rfloor} f(j) = \sum_{j=1}^M c_{\rho,n}(j) f(j). \quad (1.3)$$

For any scalar factors v_j ,

$$\prod_{(i,j) \in \mathcal{R}_{\rho,n}} v_j = \prod_{\substack{1 \leq j \leq M \\ c_{\rho,n}(j) > 0}} v_j^{c_{\rho,n}(j)}. \quad (1.4)$$

Proof. For fixed j , the allowed indices are $\lceil j^{1/\rho} \rceil \leq i \leq \min(n, j)$. Counting them proves (1.2). Sum or multiply the identical contributions in each disjoint column to obtain the other identities. $\square$

For $\rho = 2$ we abbreviate $c_n(j) = c_{2,n}(j)$ and $N_n = |\mathcal{R}_{2,n}|$. The useful endpoint identity is

$$c_n(j) = \min(n, j) - \lceil \sqrt{j} \rceil + 1 = \min(n, j) - \lfloor \sqrt{j-1} \rfloor, \quad 1 \leq j \leq n^2. \quad (1.5)$$

Both forms include perfect squares correctly. For fixed j , letting $n \rightarrow \infty$ gives

$$d_\rho(j) = j - \lceil j^{1/\rho} \rceil + 1, \quad 1 \leq d_\rho(j) \leq j. \quad (1.6)$$

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An infinite sum can therefore be regrouped under $\sum_{j \geq 1} j|f(j)| < \infty$, or under nonnegativity with an extended value.

2. LATTICE COUNTS AND SKEW DIAGRAMS

The choice $f(j) = 1$ counts integer solutions of the inequalities in (1.1). Direct row counting gives

$$|\mathcal{R}_{\rho,n}| = \sum_{i=1}^n (\lfloor i^\rho \rfloor - i + 1) \sim \frac{n^{\rho+1}}{\rho+1}. \quad (2.1)$$

The asymptotic follows by comparing $\sum_{i \leq n} i^\rho$ with its integral; the rounding error is at most n , and $\sum_{i \leq n} i = O(n^2) = o(n^{\rho+1})$.

Proposition 2.1 (Quadratic count). *For $n \geq 1$,*

$$N_n = \sum_{i=1}^n (i^2 - i + 1) = \frac{n(n^2 + 2)}{3} = \sum_{j=1}^{n^2} c_n(j). \quad (2.2)$$

Proof. Use the usual sums of the first n integers and squares. The final equality is Proposition 1.1 at $f = 1$. $\square$

There is an exact skew-diagram interpretation in the usual English convention for partitions [3, Chapter 7]. Set

$$\lambda = (\lfloor n^\rho \rfloor, \lfloor (n-1)^\rho \rfloor, \dots, 1), \quad \mu = (n-1, n-2, \dots, 1).$$

The reversed row corresponding to i contains columns $i, i+1, \dots, \lfloor i^\rho \rfloor$ of λ/μ . Thus (2.1) gives its number of cells, and (1.2) gives its column lengths. This identifies the region with a specified combinatorial shape, rather than just with an unlabeled collection of points.

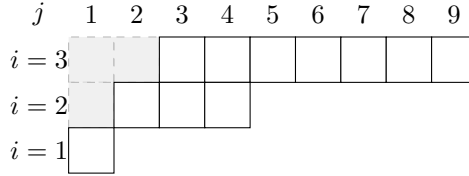


FIGURE 1. The eleven admissible cells for $n = 3$, $\rho = 2$. The dashed cells are removed from the outer partition: the skew shape is $(9, 4, 1)/(2, 1)$.

3. COLORED PARTITIONS AND OCCUPATION PRODUCTS

Assign a distinct type to each admissible pair (i, j) , and give that type size j . A configuration consists of nonnegative occupations $m_{i,j} \in \mathbb{N}_0$ with total size $\sum_{(i,j) \in \mathcal{R}_{\rho,n}} j m_{i,j}$. This is a finite-type colored-partition model within the standard weighted-partition framework [4].

Proposition 3.1 (Partition product and recurrence). *Let $p_{\rho,n}(m)$ count configurations of total size m , including $p_{\rho,n}(0) = 1$. Then*

$$P_{\rho,n}(z) = \sum_{m \geq 0} p_{\rho,n}(m) z^m = \prod_{(i,j) \in \mathcal{R}_{\rho,n}} (1 - z^j)^{-1} = \prod_{j=1}^M (1 - z^j)^{-c_{\rho,n}(j)}. \quad (3.1)$$

For $m \geq 1$,

$$m p_{\rho,n}(m) = \sum_{r=1}^m \left(\sum_{\substack{d|r \\ d \leq M}} d c_{\rho,n}(d) \right) p_{\rho,n}(m-r). \quad (3.2)$$

Proof. Each type of size j contributes its geometric series $1 + z^j + z^{2j} + \dots$. Multiplying counts all independent occupations, and (1.4) groups types of the same size. Formal logarithmic differentiation gives

$$\frac{zP'_{\rho,n}(z)}{P_{\rho,n}(z)} = \sum_{j=1}^M c_{\rho,n}(j) \frac{jz^j}{1-z^j}.$$

Expand the denominator geometrically and compare coefficients after multiplication by $P_{\rho,n}$. $\square$

For fixed degree m , the coefficients stabilize once $n \geq m$: every type of size at most m has first index at most m . Their limit has generating function

$$P_\rho(z) = \prod_{j \geq 1} (1 - z^j)^{-d_\rho(j)}, \quad |z| < 1.$$

This product also converges analytically on the disk. Indeed, on $|z| \leq r < 1$ its absolute logarithmic series is bounded by a constant times $\sum_{j \geq 1} jr^j$. Allowing each type at most once instead gives $\prod_j (1 + z^j)^{c_{\rho,n}(j)} = P_{\rho,n}(z)/P_{\rho,n}(z^2)$. These products encode specified occupation rules; the skew shape alone does not impose tableau or plane-partition monotonicity conditions on those occupations.

4. COPRIME PAIRS AND TOTIENT SUMS

The same region supports a two-index arithmetic weight:

$$C_{\rho,n} = \sum_{(i,j) \in \mathcal{R}_{\rho,n}} \mathbf{1}_{\gcd(i,j)=1}. \quad (4.1)$$

Unlike a weight depending only on j , this restriction retains information within each column. Möbius inversion gives the finite formula

$$C_{\rho,n} = \sum_{i=1}^n \sum_{d|i} \mu(d) \left(\left\lfloor \frac{\lfloor i^\rho \rfloor}{d} \right\rfloor - \left\lfloor \frac{i-1}{d} \right\rfloor \right). \quad (4.2)$$

Here μ is the Möbius function. To prove it, use $\mathbf{1}_{\gcd(i,j)=1} = \sum_{d|i, d|j} \mu(d)$ and count the multiples of d in the row interval.

Theorem 4.1 (Quadratic coprime count). *With $\varphi(1) = 1$,*

$$C_{2,n} = 1 + \sum_{i=1}^n (i-1)\varphi(i) = \frac{2}{\pi^2}n^3 + O(n^2 \log n). \quad (4.3)$$

Consequently, for a uniformly chosen pair in $\mathcal{R}_{2,n}$,

$$\mathbb{P}(\gcd(I, J) = 1) = \frac{6}{\pi^2} + O\left(\frac{\log n}{n}\right). \quad (4.4)$$

Proof. For $i > 1$, the integers $i, \dots, i^2 - 1$ consist of $i - 1$ complete residue blocks modulo i . Each contributes $\varphi(i)$ coprime values. The remaining endpoint i^2 is not coprime to i . At $i = 1$ the single pair contributes one. This proves the exact formula.

The classical estimate [5, Equation 27.11.6] $\Phi(x) = \sum_{i \leq x} \varphi(i) = 3x^2/\pi^2 + O(x \log x)$ and partial summation imply

$$\sum_{i \leq n} i\varphi(i) = n\Phi(n) - \int_1^n \Phi(x) dx = \frac{2}{\pi^2}n^3 + O(n^2 \log n).$$

Subtract $\Phi(n)$ and add one. Division by (2.2) proves (4.4). $\square$

This gives an exact arithmetic refinement of the lattice count. The density $6/\pi^2$ is the familiar coprimality constant; the finite identity specifies its realization in this curved region.

5. PROBABILITY LAWS ON ADMISSIBLE PAIRS

Choose a pair (I, J) uniformly from $\mathcal{R}_{\rho,n}$. Let $N_{\rho,n} = |\mathcal{R}_{\rho,n}|$ and $\ell_{\rho,i} = \lfloor i^\rho \rfloor - i + 1$. Then

$$\mathbb{P}(I = i) = \frac{\ell_{\rho,i}}{N_{\rho,n}}, \quad \mathbb{P}(J = j) = \frac{c_{\rho,n}(j)}{N_{\rho,n}}. \quad (5.1)$$

Conditional on $I = i$, J is uniform on the row interval. Thus choosing a row with probability $\ell_{\rho,i}/N_{\rho,n}$ and then a uniform cell samples the whole region uniformly.

For any function f on the second coordinates,

$$\mathbb{E}[f(J)] = \frac{1}{N_{\rho,n}} \sum_{j=1}^M c_{\rho,n}(j) f(j). \quad (5.2)$$

This supplies moments, distribution functions and threshold probabilities by appropriate choices of f .

Example 5.1 (A quadratic finite law). For $n = 3$ and $\rho = 2$, the multiplicities at $j = 1, \dots, 9$ are

$$(1, 1, 2, 2, 1, 1, 1, 1, 1).$$

They sum to $N_3 = 11$, and exact summation gives

$$\mathbb{E}[J] = \frac{52}{11}, \quad \mathbb{E}[J^2] = \frac{310}{11}, \quad \text{Var}(J) = \frac{706}{121}. \quad (5.3)$$

For general n in the quadratic case, direct row sums give

$$\mathbb{E}[J] = \frac{1}{2N_n} \sum_{i=1}^n (i^4 + i). \quad (5.4)$$

Indeed, the sum of j from i to i^2 is $(i^4 + i)/2$. Classical power sums therefore evaluate this mean polynomially.

6. TWO-ROOK PLACEMENTS AND COMPATIBLE ASSIGNMENTS

Regard row labels as resources and column labels as tasks, with an allowed assignment at each cell of $\mathcal{R}_{\rho,n}$. Two assignments are compatible if they use different rows and columns: they are two nonattacking rooks on the skew board [6]. Their count needs only the row and column degrees.

Proposition 6.1 (Two compatible assignments). *Let $R_2(\rho, n)$ count unordered compatible pairs and let $N = N_{\rho,n}$. Then*

$$R_2(\rho, n) = \frac{1}{2} \left(N^2 + N - \sum_{i=1}^n \ell_{\rho,i}^2 - \sum_{j=1}^M c_{\rho,n}(j)^2 \right). \quad (6.1)$$

Proof. Start with $\binom{N}{2}$ pairs of distinct cells. Exclude the $\sum_i \binom{\ell_{\rho,i}}{2}$ pairs sharing a row and the $\sum_j \binom{c_{\rho,n}(j)}{2}$ pairs sharing a column. These two excluded classes are disjoint: two distinct cells cannot share both coordinates. Expanding the binomial coefficients and using both degree sums equal to N proves (6.1). $\square$

For $n = 3$, $\rho = 2$, the row-square sum is 59 and the column-square sum is 15, so $R_2(2, 3) = 29$. The formula counts two assignments; larger rook configurations require additional intersection information.

7. RELATIVE CANONICAL PRODUCTS AND GENUS REDUCTION

Let $\rho > 1$, $\beta = 1/\rho$ and

$$r_\rho(j) = j - d_\rho(j) = \lceil j^\beta \rceil - 1.$$

The multiplicity j is the reference count, and $r_\rho(j)$ is the number removed by the power restriction. Since $r_\rho(j) \sim j^\beta$,

$$\sum_{j \geq 1} r_\rho(j) j^{-\sigma} < \infty \iff \sigma > 1 + \beta. \quad (7.1)$$

Define the standard primary factor $E_g(w) = (1 - w) \exp(\sum_{\ell=1}^g w^\ell / \ell)$. Canonical and modified determinant constructions are classical [7, 5].

Theorem 7.1 (Normalized relative product). *The products*

$$Q_0(u) = \prod_{j \geq 1} E_2(-u/j)^j, \quad Q_\rho(u) = \prod_{j \geq 1} E_2(-u/j)^{d_\rho(j)}, \quad (7.2)$$

$$R_\rho(u) = \prod_{j \geq 1} E_1(-u/j)^{r_\rho(j)}, \quad A_\rho = \sum_{j \geq 1} \frac{r_\rho(j)}{j^2} \quad (7.3)$$

define entire functions, and A_ρ is finite. Their normalizations satisfy

$$\frac{Q_0(u)}{Q_\rho(u)} = R_\rho(u) \exp(A_\rho u^2 / 2). \quad (7.4)$$

The quotient has its removable values at common zeros. The least primary-factor genus is 2 for each full spectrum and 1 for the deficit spectrum.

Proof. Uniformly on compact u -sets, $\log E_g(-u/j) = O(j^{-g-1})$ as $j \rightarrow \infty$. Thus $d_\rho(j) \sim j$ permits genus 2, while (7.1) permits genus 1 for the deficit. The corresponding lower genera fail because their leading logarithmic terms have constant sign at a fixed small positive u and divergent absolute magnitude.

At every finite level, $E_2(-u/j) = E_1(-u/j) \exp(u^2/(2j^2))$. Raise this equality to $r_\rho(j)$, multiply and pass to the locally uniform limits. This proves (7.4), initially away from zeros and then at their removable common factors. $\square$

The lower-genus product has an explicit approximation bound. If $R_{\rho,L}$ is the product through $j = L$, then for $|u| \leq B$ and $L > 2B$ its tail logarithm normalized at zero obeys

$$\left| \log \frac{R_\rho(u)}{R_{\rho,L}(u)} \right| \leq \frac{B^2 L^{\beta-1}}{2(1 - B/(L+1))(1 - \beta)}. \quad (7.5)$$

Indeed, $|\log(1 + u/j) - u/j| \leq B^2/[2j^2(1 - B/(L+1))]$ on the tail, and $r_\rho(j) \leq j^\beta$. The integral bound $\sum_{j > L} j^{\beta-2} \leq L^{\beta-1}/(1 - \beta)$ proves (7.5). For $\rho = 2$ the tail is $O(L^{-1/2})$, and the normalization constant has the positive convergent expression

$$A_2 = \sum_{k \geq 1} \zeta(2, k^2 + 1).$$

This follows from $r_2(j) = \#\{k \geq 1 : k^2 < j\}$ and Tonelli's theorem. The quadratic exponential in (7.4) retains the distinction between the two primary-factor normalizations.

8. FINITE DETERMINANTS AND GAUSSIAN NORMALIZATIONS

In the quadratic region, let L_n be the diagonal matrix on $\mathbb{R}^{N_n}$ with entry j at coordinate (i, j) . For real $u > -1$, the matrix $L_n + uI$ is positive definite.

Proposition 8.1 (Gamma evaluation and resolvent). *For $u > -1$,*

$$D_n(u) := \det(L_n + uI) = \prod_{j=1}^{n^2} (j + u)^{c_n(j)} = \prod_{i=1}^n \frac{\Gamma(i^2 + 1 + u)}{\Gamma(i + u)}, \quad (8.1)$$

$$\frac{d}{du} \log D_n(u) = \sum_{j=1}^{n^2} \frac{c_n(j)}{j + u} = \sum_{i=1}^n [\psi(i^2 + 1 + u) - \psi(i + u)]. \quad (8.2)$$

Here $\psi = \Gamma'/\Gamma$.

Proof. Count the repeated diagonal entries by columns. The gamma recurrence evaluates each row product as $\prod_{j=i}^{i^2} (j + u) = \Gamma(i^2 + 1 + u)/\Gamma(i + u)$. Logarithmic differentiation proves (8.2); the gamma formulas are classical [5, Sections 5.5 and 5.9]. $\square$

Consequently the ordinary finite Gaussian integral is

$$Z_n(u) = \int_{\mathbb{R}^{N_n}} \exp \left(-\frac{1}{2} \sum_{(i,j) \in \mathcal{R}_{2,n}} (j + u) x_{i,j}^2 \right) \prod_{(i,j)} dx_{i,j} = (2\pi)^{N_n/2} D_n(u)^{-1/2}. \quad (8.3)$$

For the resulting normalized probability density,

$$\mathbb{E}_u \left[\sum_{(i,j)} x_{i,j}^2 \right] = \text{Tr}(L_n + uI)^{-1} = \frac{d}{du} \log D_n(u).$$

Thus the product gives the normalization, while its logarithmic derivative gives a quadratic observable. At $n = 3$, $N_3 = 11$ and $D_3(0) = 4354560$. These are finite-dimensional identities. Infinite determinant comparisons use the explicit subtractions of Theorem 7.1.

9. COMPUTATIONAL VERIFICATION

Run the supplied standard-library program with Python 3.10 or later:

`python3 anc/verify_additional_applications.py`

The record is `anc/additional_applications_results.json`. Exact checks cover counts, partitions, coprimality, moments and rooks; numerical checks cover gamma formulas and relative products. The proofs establish infinite conclusions.

REFERENCES

- [1] D. Shah, M. Sahni, R. Sahni, E. León-Castro and M. Olazabal-Lugo, *Series of Floor and Ceiling Function—Part I: Partial Summations*, Mathematics **10**(7) (2022), article 1178. doi:10.3390/math10071178.
- [2] D. Shah, M. Sahni, R. Sahni, E. León-Castro and M. Olazabal-Lugo, *Series of Floor and Ceiling Functions—Part II: Infinite Series*, Mathematics **10**(9) (2022), article 1566. doi:10.3390/math10091566.
- [3] R. P. Stanley, *Enumerative Combinatorics*, Volume 2, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, 1999, Chapter 7.
- [4] B. L. Granovsky, D. Stark and M. Erlihson, *Meinardus' theorem on weighted partitions: extensions and a probabilistic proof*, Advances in Applied Mathematics **41**(3) (2008), 307–328. doi:10.1016/j.aam.2007.11.001.
- [5] NIST Digital Library of Mathematical Functions, <https://dlmf.nist.gov/>, Sections 5.5 and 5.9 (gamma and psi functions), 5.17 (Barnes functions), and 27.11 (summatory arithmetic functions). Accessed 15 October 2026.
- [6] A. J. Klein, J. B. Lewis and A. H. Morales, *Counting matrices over finite fields with support on skew Young diagrams and complements of Rothe diagrams*, Journal of Algebraic Combinatorics **39**(2) (2014), 429–456. doi:10.1007/s10801-013-0453-x.
- [7] T. Britz, A. Carey, F. Gesztesy, R. Nichols, F. Sukochev and D. Zanin, *The product formula for regularized Fredholm determinants*, Proceedings of the American Mathematical Society, Series B **8** (2021), 42–51. doi:10.1090/bproc/70.

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