

FLOOR AND CEILING LEVEL COUNTS: SUMS, PRODUCTS AND APPLICATIONS

DHAIRYA SHAH

ABSTRACT. Inverse floor and ceiling level counts provide a common measure for finite and infinite sums and products. We extend their summation framework to clipped intervals, signed weights and scalar products, with ordered convergence criteria that control incomplete blocks. The principal applications concern spherical spectral cutoffs, determinants and Gaussian integrals. The finite products have explicit gamma and Barnes evaluations; canonical normalization gives modified Fredholm determinants whose logarithmic derivatives are convergent resolvent traces and whose local coefficients are spectral zeta values. Floor–ceiling cancellation reduces the genus of a relative product, while balanced shifts yield an ordinary convergent comparison of four Gaussian systems. Explicit tail bounds control both constructions. Rational scales produce periodic corrections and finite special-function evaluations, including an odd azimuthal spherical sector. Occupation products connect the same counts with colored partitions. The determinant and special-function theory used in these evaluations is classical; the contribution is its explicit realization through rounded counts, with compatible finite approximations and normalizations. Supporting examples treat probability, analytic boundaries, Beatty allocation, integral comparisons, reciprocal-Fibonacci moments and finite-sum compression. A power-bounded double-sum model yields a theta trace, a fractional zeta pole and an explicit secondary term in a partition logarithm.

1. INTRODUCTION

A rounded index determines a counting measure on integer levels. Summing a weight over the input indices becomes a sum over these levels, and multiplying a scalar factor uses the same multiplicities as exponents. The inverse-level formulas and basic finite and infinite summation transforms are developed in [1, 2]. The present paper continues that framework through finite and infinite products and their direct spectral, partition and Gaussian applications.

The extensions beyond those summation formulas are specified as follows. Section 3 gives clipped and signed weighted identities and criteria for convergence at every input cutoff, including incomplete blocks; the finite identities use elementary grouping by fibers. Sections 4.3–4.5 connect these formulas with finite spherical determinants and quadratic means. Section 9 derives their normalized infinite limits, the relative-product genus reduction from $f_q - c_q$, and a balanced four-shift Gaussian comparison with explicit cutoff bounds. Sections 7–8 give a finite residue-based evaluation method and identify a rational-scale count with an actual spherical symmetry sector. These are explicit extensions and worked constructions of the rounded-count method.

Section 15.2 extends the finite count method to power-bounded sets of integer pairs. A quadratic cutoff gives an exact theta transform and a finite-cutoff bound. Its multiplicities yield a fractional pole in a Dirichlet transform and a fractional-power correction to an occupation-product logarithm.

The principal spectrum is the unit sphere. The ceiling count $j^2 - (j - 1)^2 = 2j - 1$ is the multiplicity of the eigenvalue j of $A = \sqrt{-\Delta_{S^2} + 1/4} + 1/2$. Its geometric transform is a one-particle trace, and C_2 is its spectral zeta function. At a finite cutoff, the sum transformation computes traces and Gaussian quadratic observables, while the product transformation computes determinants and Gaussian normalizations. Unrestricted occupations of the same

2020 *Mathematics Subject Classification.* Primary 40A05, 40A20; Secondary 05A17, 33B15, 47B10, 58J52.

Key words and phrases. Floor and ceiling functions, infinite products, spectral determinants, Gaussian integrals, weighted partitions, Barnes G -function, double sums, theta functions.

modes give colored-partition products. Thus the applications share one specified spectrum and one counting measure.

Canonical normalization gives a modified Fredholm determinant with convergent resolvent derivatives and zeta-valued Taylor coefficients. The $F_q - C_q$ cancellation removes the leading multiplicity and reduces the number of product subtractions. Balanced shifts cancel the subtractions themselves and produce an ordinary convergent comparison of finite Gaussian models. Gamma and Barnes formulas evaluate these limits, and tail bounds make their finite approximations quantitative. For $a = 1/2, b = 2$, the ceiling count selects spherical harmonics with odd azimuthal order; its periodic correction persists in the traces, occupation products and determinant responses.

The spherical spectrum [3, 5], weighted-partition construction [4], canonical and modified determinants [8, 9], and Barnes product formulas [5] are classical foundations. Earlier sphere-determinant and Barnes-zeta work [15] already links zeta values and regularized products; Section 9 makes the relation to our spectra explicit. Polynomial coefficient transforms, quasipolynomial residue formulas and Fourier filters [6] are used as systematic evaluation tools. The later probability and analytic examples illustrate the same count method; their classical ingredients are identified locally [7, 11, 13, 14].

Transformation	Direct object	Role of the same count
Finite sum	Cutoff traces, finite moments and Gaussian quadratic means	Number of inputs at each level
Finite product	Ordinary cutoff determinant and Gaussian integral	Exponent of each level factor
Infinite sum	Generating functions, zeta values and resolvent traces	Convergent transform of the level measure
Infinite product	Occupation product, modified determinant and balanced Gaussian ratio	Exponent in a proved product limit

TABLE 1. The four transformations and representative applications.

Sections 10–15 develop probability laws, natural boundaries, Beatty allocation, integral comparisons, reciprocal-Fibonacci moments and finite-sum compression. These applications use the level counts and transforms introduced in Section 2. We write $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Regrouping uses nonnegativity, absolute convergence or the explicitly proved ordering conditions. Complex powers of positive numbers use the real logarithm.

2. EXACT LEVEL COUNTS AND THEIR TRANSFORMS

The inverse-level formulas [1, 2] are stated below with both input conventions and explicit endpoint counts.

2.1. Counting both endpoint conventions. Let $a, b > 0$, put $\kappa = 1/a$, and set $T_n = (bn)^a$. Define

$$\mu_f(j; a, b) = \#\{n \in \mathbb{N}_0 : \lfloor T_n \rfloor = j\}, \quad j \in \mathbb{N}_0, \quad (2.1)$$

$$\mu_c(j; a, b) = \#\{n \in \mathbb{N} : \lceil T_n \rceil = j\}, \quad j \in \mathbb{N}. \quad (2.2)$$

The floor count includes $n = 0$, while the ceiling count does not.

Proposition 2.1 (Exact multiplicities). *For $a, b > 0$,*

$$\mu_f(j; a, b) = \left\lceil \frac{(j+1)^\kappa}{b} \right\rceil - \left\lceil \frac{j^\kappa}{b} \right\rceil, \quad j \geq 0, \quad (2.3)$$

$$\mu_c(j; a, b) = \left\lfloor \frac{j^\kappa}{b} \right\rfloor - \left\lfloor \frac{(j-1)^\kappa}{b} \right\rfloor, \quad j \geq 1. \quad (2.4)$$

For any weight w , under nonnegativity or absolute convergence,

$$\sum_{n \geq 0} w(\lfloor T_n \rfloor) = \sum_{j \geq 0} \mu_f(j; a, b) w(j), \quad (2.5)$$

$$\sum_{n \geq 1} w(\lceil T_n \rceil) = \sum_{j \geq 1} \mu_c(j; a, b) w(j). \quad (2.6)$$

Proof. The equality $\lfloor T_n \rfloor = j$ is equivalent to $j^\kappa/b \leq n < (j+1)^\kappa/b$. The number of integers in a half-open interval $[u, v)$ is $\lceil v \rceil - \lceil u \rceil$. Its lower endpoint is nonnegative here. Similarly, $\lceil T_n \rceil = j$ is equivalent to $(j-1)^\kappa/b < n \leq j^\kappa/b$. The number of integers in $(u, v]$ is $\lfloor v \rfloor - \lfloor u \rfloor$; at $j = 1$ the lower endpoint is zero, so all counted indices are positive. This proves the counts. The fibers of either rounded map are disjoint and exhaustive. Summing over those fibers proves the regroupings by Tonelli's theorem or absolute convergence. $\square$

Equations (2.3)–(2.4) are the repetition counts underlying Part I, Section 3, equations (9) and (15), and the infinite regroupings of Part II, Section 3 [1, 2]. We have restated them with full zero-level and endpoint conventions. In particular, if $b < 1$, the floor level zero can contain several indices.

2.2. Geometric and Dirichlet transforms. Define, for $|z| < 1$,

$$H_f^{(0)}(z; a, b) = \sum_{n \geq 0} z^{\lfloor (bn)^a \rfloor} = \sum_{j \geq 0} \mu_f(j; a, b) z^j, \quad (2.7)$$

$$H_f^{(+)}(z; a, b) = H_f^{(0)}(z; a, b) - 1, \quad (2.8)$$

$$H_c(z; a, b) = \sum_{n \geq 1} z^{\lceil (bn)^a \rceil} = \sum_{j \geq 1} \mu_c(j; a, b) z^j. \quad (2.9)$$

The positive-index floor series differs from the zero-index series by exactly the $n = 0$ term, even when other indices occupy level zero. Convergence follows either from the original stretched-exponential terms or from the counts, which grow at most polynomially.

Proposition 2.2 (Cumulative-count formulas). *For $|z| < 1$,*

$$H_c(z; a, b) = (1 - z) \sum_{j \geq 1} \left\lfloor \frac{j^\kappa}{b} \right\rfloor z^j, \quad (2.10)$$

$$H_f^{(0)}(z; a, b) = \frac{1 - z}{z} \sum_{j \geq 1} \left\lceil \frac{j^\kappa}{b} \right\rceil z^j. \quad (2.11)$$

The second formula is interpreted by its removable value at $z = 0$.

Proof. Let $A_j = \lfloor j^\kappa/b \rfloor$ and $B_j = \lceil j^\kappa/b \rceil$, with $A_0 = B_0 = 0$. Equations (2.3)–(2.4) say $\mu_c(j) = A_j - A_{j-1}$ and $\mu_f(j) = B_{j+1} - B_j$. Subtract the corresponding power series. Polynomial growth justifies each operation on $|z| < 1$. $\square$

For the important integer-root case, take $a = 1/q$, $b = 1$ with $q \in \mathbb{N}$, and write

$$f_q(j) = (j+1)^q - j^q, \quad c_q(j) = j^q - (j-1)^q. \quad (2.12)$$

The first is used for $j \geq 0$, the second for $j \geq 1$. The rounded geometric series are

$$H_f^{(0)}(z; 1/q, 1) = \frac{1 - z}{z} \text{Li}_{-q}(z), \quad H_{c,q}(z) := H_c(z; 1/q, 1) = (1 - z) \text{Li}_{-q}(z). \quad (2.13)$$

Here $\text{Li}_s(z) = \sum_{j \geq 1} z^j/j^s$ for $|z| < 1$. These are the geometric specializations of Part II, Corollaries 5–6, with the extra $n = 0$ floor term made explicit [2].

Define the corresponding positive-level Dirichlet series by

$$F_q(s) = \sum_{j \geq 1} f_q(j) j^{-s}, \quad C_q(s) = \sum_{j \geq 1} c_q(j) j^{-s}, \quad \text{Re } s > q. \quad (2.14)$$

The binomial theorem gives

$$F_q(s) = \sum_{k=0}^{q-1} \binom{q}{k} \zeta(s-k), \quad (2.15)$$

$$C_q(s) = \sum_{k=0}^{q-1} (-1)^{q-k+1} \binom{q}{k} \zeta(s-k). \quad (2.16)$$

These are the F/C-Shah–Riemann zeta formulas of Part II, Corollaries 3–4 [2]. Since their weights are asymptotic to qj^{q-1} , $\operatorname{Re} s > q$ is their half-plane of absolute convergence. The finite zeta expressions continue them meromorphically, with simple poles at $1, \dots, q$ and residues equal to the corresponding nonzero polynomial coefficients. This is continuation of the function, not convergence of its original series outside that half-plane.

The geometric and Dirichlet series are two transforms of the same level measure. A change of weight changes the quantity being computed without changing its multiplicities. The next sections exploit that fact in specified models.

3. FINITE AND INFINITE SUM AND PRODUCT IDENTITIES

The repeated-level counts underlying finite and infinite rounded sums [1, 2] also determine scalar products. We first state the clipped and weighted finite identities, then prove the criteria for their ordered infinite limits.

3.1. Clipped finite identities. For $1 \leq m \leq N$, let $h_f(n) = \lfloor (bn)^a \rfloor$ and $h_c(n) = \lceil (bn)^a \rceil$. Set $\kappa = 1/a$, $A_j = \lceil j^\kappa/b \rceil$ and $B_j = \lfloor j^\kappa/b \rfloor$, with $A_0 = B_0 = 0$. Write $[x]_+ = \max(x, 0)$.

Theorem 3.1 (Finite sums and finite products). *The numbers of occurrences on the positive input interval $[m, N]$ are*

$$d_{f,j}^{[m,N]} = [\min(N+1, A_{j+1}) - \max(m, A_j)]_+, \quad j \geq 0, \quad (3.1)$$

$$d_{c,j}^{[m,N]} = [\min(N, B_j) - \max(m-1, B_{j-1})]_+, \quad j \geq 1. \quad (3.2)$$

For either map h_X , $X = f, c$, and arbitrary scalar u_j, v_j ,

$$\boxed{\sum_{n=m}^N u_{h_X(n)} = \sum_j d_{X,j}^{[m,N]} u_j}, \quad (3.3)$$

$$\boxed{\prod_{n=m}^N v_{h_X(n)} = \prod_{j: d_{X,j}^{[m,N]} > 0} v_j^{d_{X,j}^{[m,N]}}}. \quad (3.4)$$

The product includes only occupied levels, so zero factors cause no 0^0 convention. The same identities hold for any map on a finite index set, with its corresponding fiber sizes.

Proof. The floor fiber is $[A_j, A_{j+1}) \cap \mathbb{Z}$ and the ceiling fiber is $(B_{j-1}, B_j] \cap \mathbb{Z}$. Intersect each with $[m, N] \cap \mathbb{Z}$ and count integers. This proves the two clipped formulas even for an empty intersection or a skipped level. The fibers partition the finite input set. Add the constant value u_j on a fiber, or multiply its d_j copies of the scalar v_j , to obtain the identities. $\square$

These formulas account for both partially cut end blocks. For example, on $n = 7, \dots, 18$, $\lfloor \sqrt{n} \rfloor$ has counts 2, 7, 3 at levels 2, 3, 4. Thus its sum is 37, and its product is $2^2 3^7 4^3 = 559872$. At $b < 1$ the positive-input floor level zero contains $A_1 - 1$ indices, not A_1 ; the latter includes $n = 0$.

Proposition 3.2 (Weights, restrictions and composition). *Let S be finite, $h : S \rightarrow J$ any map, and $W_j = \sum_{n \in S: h(n)=j} \omega_n$. Then*

$$\sum_{n \in S} \omega_n u_{h(n)} = \sum_j W_j u_j. \quad (3.5)$$

For integer weights and nonzero factors,

$$\prod_{n \in S} v_{h(n)}^{\omega_n} = \prod_{j: W_j \neq 0} v_j^{W_j}. \quad (3.6)$$

For arbitrary complex weights the last identity instead means exponentiation of the weighted sum of specified logarithms.

Proof. Sum the weights on each disjoint fiber. The additive identity follows by distributivity; the multiplicative identity follows from the laws of integer powers of nonzero scalars. With complex exponents, define each power by the chosen logarithm and use the additive identity. $\square$

For a clipped nonempty fiber $[L, U) \cap \mathbb{Z}$ and the input prefix $\Phi(t) = \sum_{n=m}^t \omega_n$, its weight is $\Phi(U-1) - \Phi(L-1)$, with $\Phi(t) = 0$ for $t < m$. Restricting inputs to $n \equiv r \pmod{d}$ replaces its count by $\lceil (U-r)/d \rceil - \lceil (L-r)/d \rceil$. Empty fibers contribute zero. If a further map g groups the output labels, its weights are $\sum_{j: g(j)=\ell} W_j$. Thus finite nested rounded transformations can be performed successively without discarding their counts.

3.2. Ordered infinite sums and products. A finite identity alone does not justify an infinite limit. For a nondecreasing unbounded map $h : \mathbb{N} \rightarrow \mathbb{N}_0$, every nonempty fiber is a finite consecutive block. Let d_j be its size. All level sums below use increasing occupied j .

Theorem 3.3 (The infinite sum and product identities). *The ordered series $\sum_{n \geq 1} u_{h(n)}$ converges if and only if the ordered level series $\sum_j d_j u_j$ converges. In that case,*

$$\boxed{\sum_{n \geq 1} u_{h(n)} = \sum_j d_j u_j}. \quad (3.7)$$

Absolute convergence is equivalent to $\sum_j d_j |u_j| < \infty$.

If all occupied factors v_j are nonzero, the ordered product has a finite nonzero limit if and only if $\sum_j d_j \text{Log } v_j$ converges, taking logarithms near one eventually. Its value is

$$\boxed{\prod_{n \geq 1} v_{h(n)} = \exp \left(\sum_j d_j \text{Log } v_j \right)}. \quad (3.8)$$

Finitely many initial logarithm choices change the exponent by a multiple of $2\pi i$ and do not change the value. The sufficient absolute condition is $\sum_j d_j |v_j - 1| < \infty$.

Proof. Complete-block sum endpoints are a subsequence of the original partial sums. Conversely, convergence of $\sum_j d_j u_j$ implies $d_j u_j \rightarrow 0$ on occupied levels. A partial sum within that block differs from the preceding endpoint by tu_j , $0 \leq t \leq d_j$, of modulus at most $|d_j u_j| \rightarrow 0$. This proves equality of all ordered limits. Nonnegative grouping proves the absolute statement.

If the logarithm series converges, then $d_j \text{Log } v_j \rightarrow 0$ and the remaining logarithm inside a block has modulus at most $|d_j \text{Log } v_j|$. All partial products therefore tend to the exponential of the same sum. Conversely, suppose partial products $P_n \rightarrow P \neq 0$. Eventually they lie in a disk about P excluding zero, with an analytic logarithm ℓ . The factors P_n/P_{n-1} tend to one and, eventually, $\text{Log } v_{h(n)} = \ell(P_n) - \ell(P_{n-1})$. These logarithms telescope to a convergent series. Apply the sum statement to its identical block terms. Finally $|\text{Log } v_j| \asymp |v_j - 1|$ near one, proving the absolute criterion. $\square$

For h_c its d_j are $\mu_c(j; a, b)$. For the positive-input floor map, $d_0 = \lceil 1/b \rceil - 1$ and $d_j = \mu_f(j; a, b)$ for $j \geq 1$. This distinction retains the earlier zero-level convention.

For example, $h(n) = \lceil \sqrt{n} \rceil$ and $u_j = (-1)^{j+1}/(j(2j-1))$ give the conditionally convergent identities

$$\sum_{n \geq 1} \frac{(-1)^{h(n)+1}}{h(n)(2h(n)-1)} = \log 2, \quad \prod_{n \geq 1} \exp \left[\frac{(-1)^{h(n)+1}}{h(n)(2h(n)-1)} \right] = 2. \quad (3.9)$$

The grouped logarithm is the classical alternating harmonic series. The examples illustrate the ordering theorem; the substantial product applications below are determinants and occupation models.

3.3. Incomplete blocks and convergence. For products, observing complete-block endpoints is insufficient. Take $h(n) = \lceil \sqrt{n} \rceil$, $d_j = 2j - 1$ and $v_j = e^{2\pi i/(2j-1)}$. Every full block product is one, and $v_j \rightarrow 1$, but the product after $j - 1$ entries of a late block tends to -1 . The original partial products have no limit.

More generally, if the fiber with label j starts at input a_j and has size d_j , define

$$U_j(t) = \sum_{n=a_j}^{a_j+t-1} \omega_n, \quad M_j = \max_{0 \leq t \leq d_j} |U_j(t)|.$$

Here t is an integer and $U_j(0) = 0$. An ordered weighted sum converges precisely when its endpoint series converges and $|u_j|M_j \rightarrow 0$. For integer weights and nonzero factors, a nonzero weighted product limit requires and is implied by convergence of its endpoint products to $P \neq 0$ and $\max_t |v_j^{U_j(t)} - 1| \rightarrow 0$. Indeed, an intermediate sum is the preceding endpoint plus $u_j U_j(t)$; an intermediate product is that endpoint times $v_j^{U_j(t)}$. These two expressions prove both criteria.

The four identities use the same fibers for additive and multiplicative transformations. Their infinite versions retain the input order and explicitly control incomplete blocks. The next sections specify what their weights and factors compute: one-particle traces, finite spectral determinants, Gaussian integrals and many-particle occupation products.

4. SPECTRAL GEOMETRY ON THE SPHERE

The spherical harmonic spectrum [3, 5] supplies the rounded multiplicities. Sphere determinants and Barnes-zeta methods have an established literature [15]; the operators A and B below are first-order spectral functions of the Laplacian.

4.1. The full spherical spectrum. On the unit sphere S^2 , the nonnegative Laplace operator $\mathcal{L} = -\Delta_{S^2}$ has eigenvalues $\ell(\ell+1)$ of multiplicity $2\ell+1$, $\ell \in \mathbb{N}_0$. The eigenspaces are the spherical harmonics of degree ℓ [3, 5].

Theorem 4.1 (Rounded enumeration of spherical modes). *The spectrum of $\mathcal{L}$, counted with multiplicity and indexed from $n = 0$, is*

$$\lambda_n = \lfloor \sqrt{n} \rfloor (\lfloor \sqrt{n} \rfloor + 1). \quad (4.1)$$

For a nonnegative spectral weight g , or a complex weight with $\sum_{n \geq 0} |g(\lambda_n)| < \infty$,

$$\text{Tr } g(\mathcal{L}) = \sum_{n \geq 0} g(\lambda_n) = \sum_{\ell \geq 0} (2\ell+1)g(\ell(\ell+1)). \quad (4.2)$$

For nonnegative g , this equality may be interpreted with infinite sums.

Proof. The integers with $\lfloor \sqrt{n} \rfloor = \ell$ are exactly $\ell^2, \ell^2 + 1, \dots, (\ell+1)^2 - 1$. There are $(\ell+1)^2 - \ell^2 = 2\ell+1$ such indices. Thus each prescribed eigenvalue has exactly its spherical multiplicity. Apply Proposition 2.1 to the weight $w(\ell) = g(\ell(\ell+1))$. $\square$

For example, for $\tau > 0$ the heat trace is

$$K(\tau) = \text{Tr } e^{-\tau \mathcal{L}} = \sum_{n \geq 0} e^{-\tau \lfloor \sqrt{n} \rfloor (\lfloor \sqrt{n} \rfloor + 1)}. \quad (4.3)$$

The regrouped form converges because its summands are bounded by a polynomial times a Gaussian in ℓ . For $\nu > 0$, the massive spectral zeta function

$$Z_\nu(s) = \sum_{n \geq 0} (\lambda_n + \nu^2)^{-s} = \sum_{\ell \geq 0} (2\ell + 1)(\ell(\ell + 1) + \nu^2)^{-s} \quad (4.4)$$

converges absolutely for $\text{Re } s > 1$. This follows by comparing its large- ℓ terms with $\ell^{1-2\text{Re } s}$. For a sphere of radius R , replace $\mathcal{L}$ by $R^{-2}\mathcal{L}$ in these formulas.

Corollary 4.2 (Exact spectral count). *For $\Lambda \geq 0$, the number of eigenvalues of $\mathcal{L}$ at most Λ , counting the zero mode and multiplicities, is*

$$N_{\mathcal{L}}(\Lambda) = (L + 1)^2, \quad L = \left\lfloor \frac{\sqrt{1 + 4\Lambda} - 1}{2} \right\rfloor. \quad (4.5)$$

Proof. The admissible degrees are $0, \dots, L$. Their multiplicities sum to $\sum_{\ell=0}^L (2\ell + 1) = (L + 1)^2$. $\square$

4.2. Rounded functions as traces of spectral square roots. Functional calculus for $\mathcal{L}$ defines

$$A = \sqrt{\mathcal{L} + \frac{1}{4}} + \frac{1}{2}, \quad B = \sqrt{\mathcal{L} + \frac{1}{4}} - \frac{1}{2}. \quad (4.6)$$

On degree- ℓ harmonics, A has eigenvalue $\ell + 1$ and B has eigenvalue ℓ .

Proposition 4.3 (Geometric and zeta traces). *For $\tau > 0$,*

$$\text{Tr } e^{-\tau A} = H_{c,2}(e^{-\tau}) = \frac{e^{-\tau}(1 + e^{-\tau})}{(1 - e^{-\tau})^2}, \quad (4.7)$$

$$\text{Tr } e^{-\tau B} = H_f^{(0)}(e^{-\tau}; 1/2, 1) = \frac{1 + e^{-\tau}}{(1 - e^{-\tau})^2}. \quad (4.8)$$

For $\text{Re } s > 2$,

$$\zeta_A(s) = 2\zeta(s - 1) - \zeta(s), \quad \zeta_B^*(s) = 2\zeta(s - 1) + \zeta(s), \quad (4.9)$$

where the star omits the zero eigenvalue of B .

Proof. After setting $j = \ell + 1$, the eigenvalue j of A has multiplicity $2j - 1 = c_2(j)$. The positive eigenvalue j of B has multiplicity $2j + 1 = f_2(j)$, and its zero eigenvalue has multiplicity one. This proves the trace and Dirichlet-series identifications. Sum $\sum_{j \geq 1} z^j = z/(1 - z)$ and $\sum_{j \geq 1} jz^j = z/(1 - z)^2$ to obtain the rational expressions. The zeta formulas follow from (2.15)–(2.16). $\square$

As a further consequence, the spectral determinant of ϵA , $\epsilon > 0$, can be defined by the regular continuation at zero:

$$\det_\zeta(\epsilon A) := \exp \left[- \frac{d}{ds} (\epsilon^{-s} C_2(s)) \Big|_{s=0} \right]. \quad (4.10)$$

Using $\zeta(0) = -1/2$, $\zeta(-1) = -1/12$ and $\zeta'(0) = -\frac{1}{2} \log(2\pi)$ [5], $C_2(0) = 1/3$, so

$$\det_\zeta(\epsilon A) = \epsilon^{1/3} \frac{\exp[-2\zeta'(-1)]}{\sqrt{2\pi}}. \quad (4.11)$$

This quantity is a zeta-regularized determinant, rather than an ordinarily convergent product of eigenvalues.

The square-root floor count encodes the actual multiplicities of a familiar geometry. The geometric series itself is the trace of a first-order spectral function, while the Laplace heat trace uses the different weight $\ell \mapsto e^{-\tau \ell(\ell+1)}$. It is essential to distinguish these two operators.

For the round sphere all formulas are exact. For a different metric, the multiplicities and eigenvalues need a separate analysis; the rounded enumeration above should not be assumed to persist.

4.3. Finite sums: spectral cutoffs and quadratic observables. Let $\mathcal{H}_M$ contain all spherical harmonics with $\ell = 0, \dots, M-1$. Its dimension is M^2 . The restriction $A_M = A|_{\mathcal{H}_M}$ has eigenvalue j with multiplicity $2j-1$, $1 \leq j \leq M$. The square-root ceiling count therefore gives, for any g ,

$$\mathrm{Tr} g(A_M) = \sum_{n=1}^{M^2} g(\lceil \sqrt{n} \rceil) = \sum_{j=1}^M (2j-1)g(j). \quad (4.12)$$

This is exactly the finite sum transformation (3.3), now acting on an actual spectral cutoff.

For $t > 0$, put $r = e^{-t}$. The finite one-particle trace is $K_M(t) = \sum_{j=1}^M (2j-1)r^j$. Its difference from the infinite trace is the exact positive tail

$$\mathrm{Tr} e^{-tA} - K_M(t) = r^{M+1} \left[\frac{2M+1}{1-r} + \frac{2r}{(1-r)^2} \right]. \quad (4.13)$$

To derive it, write $j = M+1+k$ and sum the geometric series and its first derivative. Thus the finite formula is also a way to compute the infinite trace with an explicit cutoff error.

For real $z > -1$, the cutoff resolvent trace is

$$\begin{aligned} R_M(z) &= \mathrm{Tr}(A_M + zI)^{-1} = \sum_{j=1}^M \frac{2j-1}{j+z} \\ &= 2M - (2z+1)[\psi(M+1+z) - \psi(1+z)], \end{aligned} \quad (4.14)$$

where $\psi = \Gamma'/\Gamma$. Indeed, $(2j-1)/(j+z) = 2 - (2z+1)/(j+z)$, and the logarithmic derivative of $\Gamma(x+1) = x\Gamma(x)$ gives the digamma difference. All sums here are finite; there is no analytic regularization.

4.4. Finite products: determinants and a Barnes formula. Define the ordinary determinant

$$D_M(z) = \det(A_M + zI) = \prod_{n=1}^{M^2} (\lceil \sqrt{n} \rceil + z) = \prod_{j=1}^M (j+z)^{2j-1}. \quad (4.15)$$

The last equality is the finite product transformation (3.4). It identifies the product with a finite matrix determinant, including each eigenvalue's correct multiplicity.

Proposition 4.4 (Exact finite determinant). *For $\mathrm{Re} z > -1$,*

$$D_M(z) = \frac{\Gamma(M+1+z)^{2M-1} \Gamma(1+z) G(1+z)^2}{G(M+1+z)^2}, \quad (4.16)$$

where G is Barnes' function. The finite product on the left continues the expression across its removable singularities.

Proof. Use the classical recurrences $G(x+1) = \Gamma(x)G(x)$ and $\Gamma(x+1) = x\Gamma(x)$ [5, Section 5.17]. Telescoping the gamma recurrence gives $\prod_{j=1}^M (j+z) = \Gamma(M+1+z)/\Gamma(1+z)$. A second telescoping calculation gives

$$\prod_{j=1}^M (j+z)^j = \frac{\Gamma(M+1+z)^M G(1+z)}{G(M+1+z)}.$$

Square this expression and divide by the first product, because $(j+z)^{2j-1} = ((j+z)^j)^2/(j+z)$. The result is (4.16). $\square$

For comparison, the positive part of B with degrees $1, \dots, M$ has multiplicities $2j + 1$ and determinant

$$\prod_{j=1}^M (j+z)^{2j+1} = \frac{\Gamma(M+1+z)^{2M+1} G(1+z)^2}{\Gamma(1+z) G(M+1+z)^2}. \quad (4.17)$$

Multiply, rather than divide, the same first product to prove this. For a general input interval, the clipped counts in (3.1)–(3.2) retain the two partial endpoint blocks. The complete-cutoff determinant admits the gamma and Barnes evaluation above; clipped counts handle partially occupied levels.

4.5. Finite Gaussian integrals and quadratic means. Choose real coordinates $x_{j,r}$ on $\mathcal{H}_M$, $1 \leq r \leq 2j - 1$, and let $z > -1$. The finite Gaussian integral is

$$\mathcal{Z}_M(z) = \int_{\mathbb{R}^{M^2}} \exp \left[-\frac{1}{2} \sum_{j=1}^M \sum_{r=1}^{2j-1} (j+z) x_{j,r}^2 \right] dx. \quad (4.18)$$

Each coordinate integral is $(2\pi/(j+z))^{1/2}$. Hence

$$\mathcal{Z}_M(z) = (2\pi)^{M^2/2} D_M(z)^{-1/2}, \quad \frac{\mathcal{Z}_M(z)}{\mathcal{Z}_M(0)} = \prod_{j=1}^M \left(\frac{j}{j+z} \right)^{(2j-1)/2}. \quad (4.19)$$

With the probability density normalized by this integral, each coordinate has variance $(j+z)^{-1}$. Therefore

$$\mathbb{E}_z \left[\sum_{j,r} x_{j,r}^2 \right] = R_M(z) = -2 \frac{d}{dz} \log \mathcal{Z}_M(z). \quad (4.20)$$

The derivative can also be obtained by differentiating the finite determinant. Thus the finite product gives the normalization of a specified Gaussian system, and the finite sum gives its mean quadratic observable. These are two uses of the same rounded measure.

This is an exact finite model, useful for spectral cutoffs and normalization calculations. It replaces M^2 scalar mode factors by M level factors. The infinite determinant limit requires the additional convergence analysis in Section 9; it does not follow by replacing M with infinity in (4.15).

5. PARTITIONS AND IDEAL OCCUPATION MODELS

Euler products, logarithmic recurrences and the power/Dirichlet transform pair for colored partitions are developed in [4]. We apply these constructions to rounded level counts and to the spherical spectrum.

5.1. A colored-partition model from ceiling counts. Fix $q \in \mathbb{N}$ and let $c_q(j) = j^q - (j-1)^q$. One interpretation of this integer is the number of tuples $(m_1, \dots, m_q) \in \mathbb{N}^q$ with $\max_i m_i = j$: there are j^q tuples with maximum at most j and $(j-1)^q$ with maximum at most $j-1$.

A part of size j will have $c_q(j)$ colors, each of which can occur any number of times. Let $p_q(N)$ count such partitions of N , including the empty partition at $N = 0$. The standard weighted-partition construction [4] gives

$$P_q(z) := \sum_{N \geq 0} p_q(N) z^N = \prod_{j \geq 1} (1 - z^j)^{-c_q(j)}, \quad |z| < 1. \quad (5.1)$$

The product defines a nonzero analytic function on the disk. Indeed, the logarithmic factors are bounded on each compact subdisk by a constant times $c_q(j)|z|^j$, which is summable.

Theorem 5.1 (Logarithm and coefficient recurrence). *With the logarithm normalized by $\log P_q(0) = 0$,*

$$\log P_q(z) = \sum_{r \geq 1} \frac{H_{c,q}(z^r)}{r}. \quad (5.2)$$

If $b_q(k) = \sum_{d|k} d c_q(d)$, then

$$N p_q(N) = \sum_{k=1}^N b_q(k) p_q(N-k), \quad p_q(0) = 1. \quad (5.3)$$

Proof. Expand each logarithmic factor using $-\log(1-z^j) = \sum_{r \geq 1} z^{jr}/r$. For $|z| < 1$ the double series is absolutely convergent, since $\sum_j c_q(j) |z|^j / (1-|z|^j) < \infty$. Its regrouping by r is (5.2). Next,

$$z \frac{P'_q(z)}{P_q(z)} = \sum_{j \geq 1} \frac{j c_q(j) z^j}{1-z^j} = \sum_{k \geq 1} b_q(k) z^k.$$

Multiply by $P_q(z)$ and compare coefficients. $\square$

Example 5.2 (The square-root case). Let $\mathcal{P}(z) = \prod_{j \geq 1} (1-z^j)^{-1}$ be the ordinary-partition function and $\mathcal{M}(z) = \prod_{j \geq 1} (1-z^j)^{-j}$ be MacMahon's plane-partition generating function [4]. Since $c_2(j) = 2j-1$,

$$P_2(z) = \frac{\mathcal{M}(z)^2}{\mathcal{P}(z)}. \quad (5.4)$$

Its coefficients through degree ten are

$$1, 1, 4, 9, 22, 46, 103, 208, 431, 849, 1671.$$

For example, at size two there is one partition $1+1$ and three colors of the part 2, giving $p_2(2) = 4$. With positive floor weights $f_2(j) = 2j+1$, the product instead equals $\mathcal{M}(z)^2 \mathcal{P}(z)$.

5.2. Finite occupation products and their infinite limit. For $q, M \in \mathbb{N}$, restrict the one-particle states to rounded inputs $n = 1, \dots, M^q$. For $0 \leq y \leq 1$ and $0 < z < 1$, the finite product continuation gives

$$\Xi_{q,M}(y, z) = \prod_{n=1}^{M^q} \frac{1}{1 - yz^{\lceil n^{1/q} \rceil}} = \prod_{j=1}^M (1 - yz^j)^{-c_q(j)}. \quad (5.5)$$

Its finite logarithm is $\sum_{r \geq 1} y^r H_{c,q,M}(z^r)/r$, where $H_{c,q,M}(z) = \sum_{j=1}^M c_q(j) z^j$. Expand each of the finitely many logarithms to prove this. For the infinite product, its scalar factors satisfy

$$\sum_{j \geq 1} c_q(j) \left| (1 - yz^j)^{-1} - 1 \right| \leq \frac{y}{1-yz} \sum_{j \geq 1} c_q(j) z^j < \infty.$$

Theorem 3.3 therefore proves a nonzero product limit. Its logarithm is the series in (5.6). At $y = 1$, this is $P_q(z)$; at $q = 2$ it is the unrestricted occupation model of the spherical A modes. The finite formula and the convergence theorem thus supply the finite and infinite product forms of the existing partition application.

5.3. Bosons, fermions and the one-particle trace. Consider independent modes of energy $j\epsilon$, $\epsilon > 0$, with $c_q(j)$ modes at that level, inverse temperature $\beta > 0$, and fugacity $0 \leq y \leq 1$. Put $z = e^{-\beta\epsilon}$. The bosonic grand partition function is

$$\Xi_q(y, z) = \prod_{j \geq 1} (1 - yz^j)^{-c_q(j)}, \quad \log \Xi_q(y, z) = \sum_{r \geq 1} \frac{y^r H_{c,q}(z^r)}{r}. \quad (5.6)$$

Each factor is the geometric sum over the occupation of one mode. With fermionic occupations zero or one, the analogous formula is

$$\Xi_q^F(y, z) = \prod_{j \geq 1} (1 + yz^j)^{c_q(j)}, \quad \log \Xi_q^F(y, z) = \sum_{r \geq 1} \frac{(-1)^{r+1} y^r H_{c,q}(z^r)}{r}. \quad (5.7)$$

Absolute convergence follows as in the proof of Theorem 5.1. Termwise differentiation on compact parameter intervals gives the mean bosonic particle number and energy:

$$\mathbb{E}[\mathbf{N}] = \sum_{j \geq 1} c_q(j) \frac{yz^j}{1 - yz^j}, \quad (5.8)$$

$$\mathbb{E}[\mathbf{E}] = \epsilon \sum_{j \geq 1} j c_q(j) \frac{yz^j}{1 - yz^j}. \quad (5.9)$$

For $q = 2$, the one-particle Hamiltonian can be chosen as ϵA from (4.6), so these multiplicities have the concrete spherical realization already established.

5.4. Zeta transforms in a Mellin formula.

Theorem 5.3 (Mellin representation and leading growth). *For $t > 0$ and any $c > q$,*

$$\log P_q(e^{-t}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \zeta(s+1) C_q(s) t^{-s} ds. \quad (5.10)$$

Furthermore,

$$\log P_q(e^{-t}) \sim q! \zeta(q+1) t^{-q} \quad \text{as } t \downarrow 0. \quad (5.11)$$

Proof. Mellin inversion of the exponential gives $e^{-x} = (2\pi i)^{-1} \int_{\text{Re } s=c} \Gamma(s) x^{-s} ds$ for $x > 0, c > 0$ [5, Sections 2.5 and 5.11]. Insert this in $\log P_q(e^{-t}) = \sum_{j,r \geq 1} c_q(j) e^{-tjr} / r$. On the line $\text{Re } s = c > q$, the sum of absolute values is bounded by

$$t^{-c} \left(\sum_{j \geq 1} c_q(j) j^{-c} \right) \left(\sum_{r \geq 1} r^{-c-1} \right) \int_{-\infty}^{\infty} |\Gamma(c+iu)| du.$$

This is finite: the first two sums converge and the gamma factor has exponential decay along the vertical line. Fubini's theorem proves (5.10).

For the asymptotic assertion, a direct positive argument avoids contour shifts. The polynomial $c_q(j)$ has degree $q-1$ and leading coefficient q . Riemann sums give

$$\lim_{t \downarrow 0} t^q H_{c,q}(e^{-t}) = q \int_0^{\infty} x^{q-1} e^{-x} dx = q!.$$

For each lower-degree term its correspondingly scaled sum tends to zero. The function $u \mapsto u^q H_{c,q}(e^{-u})$ is bounded on $u > 0$: it has a finite limit at zero, is continuous in between, and tends to zero at infinity. Thus, for some K_q ,

$$\frac{t^q H_{c,q}(e^{-rt})}{r} \leq K_q r^{-q-1}.$$

Use (5.2) and dominated convergence to obtain $q! \sum_{r \geq 1} r^{-q-1}$. $\square$

$H_{c,q}$ is the one-particle level generating function; C_q is the Dirichlet transform of the same weights and enters the Mellin integral. For $q = 2$ the leading logarithmic growth is $2\zeta(3)t^{-2}$. The asymptotics of the individual coefficients $p_q(N)$ require additional coefficient-extraction analysis. In particular, the single-positive-pole form of Meinardus' theorem should not be invoked without modification for a Dirichlet series with several positive poles. Also, for $q \geq 3$, the product (5.1) counts the specified colored partitions; it is not automatically a formula for solid partitions.

6. A FINITE EVALUATION METHOD FOR COEFFICIENT REPETITION

The square-root Maclaurin identity of [2, Section 11.5] extends to integer roots through the Euler differential operator.

Let $f(x) = \sum_{j \geq 0} a_j x^j$ have radius of convergence $R > 0$, and put $\mathcal{D} = x d/dx$. For $q \in \mathbb{N}$, define the two repeated-coefficient transforms

$$(\mathcal{T}_{f,q}f)(x) = \sum_{n \geq 0} a_{\lfloor n^{1/q} \rfloor} x^{\lfloor n^{1/q} \rfloor}, \quad (6.1)$$

$$(\mathcal{T}_{c,q}f)(x) = \sum_{n \geq 1} a_{\lceil n^{1/q} \rceil} x^{\lceil n^{1/q} \rceil}. \quad (6.2)$$

Theorem 6.1 (Finite differential representation). *For $|x| < R$,*

$$\mathcal{T}_{f,q}f = ((\mathcal{D} + 1)^q - \mathcal{D}^q)f, \quad (6.3)$$

$$\mathcal{T}_{c,q}f = (\mathcal{D}^q - (\mathcal{D} - 1)^q)(f - a_0). \quad (6.4)$$

Both operators have order $q - 1$. Their transformed series have the same radius of convergence as f when f has infinitely many nonzero coefficients. Wherever f has a meromorphic continuation, these formulas give meromorphic continuations of its transforms, with no new singular locations.

Proof. The coefficient at x^j in the first transform is $f_q(j)a_j$, while in the second it is $c_q(j)a_j$ for $j \geq 1$. The operator $\mathcal{D}$ acts on x^j by multiplication by j , giving the two polynomials in (6.3)–(6.4). Their leading degree- q terms cancel, leaving degree $q - 1$, with leading coefficient q .

For $|x| < R$, choose r with $|x| < r < R$. Absolute convergence at r , together with the polynomial growth of $f_q(j)$ and $c_q(j)$, justifies the regrouping and all termwise derivatives. Finally, $f_q(j)^{1/j} \rightarrow 1$ and $c_q(j)^{1/j} \rightarrow 1$. Cauchy–Hadamard therefore gives the same radius. Subtracting the constant does not affect it. Finite differentiation and multiplication by polynomials preserve meromorphic continuation and analyticity at every originally analytic point. $\square$

The first cases are

$$\mathcal{T}_{f,2}f = f + 2xf', \quad \mathcal{T}_{c,2}f = 2xf' - (f - a_0), \quad (6.5)$$

$$\mathcal{T}_{f,3}f = f + 6xf' + 3x^2f'', \quad \mathcal{T}_{c,3}f = 3x^2f'' + (f - a_0). \quad (6.6)$$

The square-root floor case is the Maclaurin-series identity given in Part II, Section 11.5 [2]. The operator form gives its systematic integer-root extension.

Example 6.2 (Elementary closed forms). Taking $f(x) = e^x$ in (6.6) gives

$$\sum_{n \geq 0} \frac{x^{\lfloor n^{1/3} \rfloor}}{\lfloor n^{1/3} \rfloor!} = (3x^2 + 6x + 1)e^x. \quad (6.7)$$

For $f(x) = \sin x$ the square-root transform is $\sin x + 2x \cos x$. These are identities of entire functions. Taking $f(x) = (1 - x)^{-1}$ and $q = 2$ gives $\mathcal{T}_{f,2}f = (1 + x)/(1 - x)^2$ and $\mathcal{T}_{c,2}f = x(1 + x)/(1 - x)^2$. Thus the same derivative rule evaluates the spherical one-particle traces in (4.7)–(4.8); it is not a separate application family in the present narrative.

Corollary 6.3 (Polylogarithm transforms). *For $|x| < 1$ and $s \in \mathbb{C}$,*

$$\mathcal{T}_{f,q} \text{Li}_s(x) = \sum_{k=0}^{q-1} \binom{q}{k} \text{Li}_{s-k}(x), \quad (6.8)$$

$$\mathcal{T}_{c,q} \text{Li}_s(x) = \sum_{k=0}^{q-1} (-1)^{q-k+1} \binom{q}{k} \text{Li}_{s-k}(x). \quad (6.9)$$

At $x = 1$ these become $F_q(s)$ and $C_q(s)$ when $\operatorname{Re} s > q$.

Proof. The defining coefficients of Li_s are $a_0 = 0$ and $a_j = j^{-s}$ for $j \geq 1$ [5, Section 25.12]. Multiply them by the two binomial polynomials. At $x = 1$ absolute convergence holds exactly in the specified half-plane. $\square$

This is a calculus for coefficient repetition. It replaces an infinite-looking rounded-index construction by finitely many derivatives of the original analytic function. The order is $q - 1$, so its evaluation cost depends on the chosen root order and on access to derivatives of f . The representation does not imply that singularities must remain: cancellation can remove a singularity, although it cannot create one at an originally analytic point.

7. RATIONAL SCALES AND FINITE SPECIAL-FUNCTION FORMULAS

At integer roots, rational scales give quasipolynomial multiplicities. Residue decomposition and finite Fourier analysis [6] then yield explicit formulas for their transforms.

The repetition patterns investigated in Part II, Section 10 [2], can be treated systematically by residue classes. We give the argument at every positive rational scale. Let $a = 1/q$ with $q \in \mathbb{N}$, and write $b = u/v$ with $u, v \in \mathbb{N}$ relatively prime.

Theorem 7.1 (Periodic corrections). *Let r_j be the least nonnegative residue of vj^q modulo u , and let d_j be the least nonnegative residue of $-vj^q$ modulo u . Then both sequences are u -periodic (the period need not be minimal), and*

$$\mu_c(j; 1/q, u/v) = \frac{v c_q(j) - r_j + r_{j-1}}{u}, \quad j \geq 1, \quad (7.1)$$

$$\mu_f(j; 1/q, u/v) = \frac{v f_q(j) + d_{j+1} - d_j}{u}, \quad j \geq 0. \quad (7.2)$$

Consequently the multiplicities are quasipolynomials: on each residue class modulo u , they are polynomials of degree at most $q - 1$, allowing the zero polynomial.

Proof. By Euclidean division, $\lfloor vj^q/u \rfloor = (vj^q - r_j)/u$ and $\lceil vj^q/u \rceil = (vj^q + d_j)/u$. Substitute these into (2.4) and (2.3). The congruence $(j + u)^q \equiv j^q \pmod{u}$ proves periodicity. A periodic correction is constant on each residue class, proving the final assertion. For $q \geq 2$ the leading coefficient is $vq/u > 0$, but for $q = 1$ a constituent can vanish: at $b = 2$, $\mu_c(j; 1, 2)$ is zero for odd j and one for even j . This is why the degree statement allows zero constituents. $\square$

Set

$$R_{q,u,v}(z) = \sum_{\ell=1}^u r_\ell z^\ell, \quad D_{q,u,v}(z) = \sum_{\ell=1}^u d_\ell z^\ell. \quad (7.3)$$

The endpoint residues r_u, d_u are both zero.

Corollary 7.2 (Rational geometric functions). *For $|z| < 1$,*

$$H_c(z; 1/q, u/v) = \frac{1-z}{u} \left[v \operatorname{Li}_{-q}(z) - \frac{R_{q,u,v}(z)}{1-z^u} \right], \quad (7.4)$$

$$H_f^{(0)}(z; 1/q, u/v) = \frac{1-z}{uz} \left[v \operatorname{Li}_{-q}(z) + \frac{D_{q,u,v}(z)}{1-z^u} \right]. \quad (7.5)$$

Both are rational functions. Their possible poles are u -th roots of unity, and a common denominator divides $(1 - z^u)^q$.

Proof. Use Proposition 2.2 and $\sum_{j \geq 1} r_j z^j = R_{q,u,v}(z)/(1 - z^u)$, with the analogous formula for d_j . The function $\operatorname{Li}_{-q}(z) = \mathcal{D}^q(z/(1-z))$ is rational with denominator dividing $(1-z)^{q+1}$. The prefactor $1-z$ leaves at most order q there. The periodic part has denominator $1-z^u$. These denominators both divide $(1-z^u)^q$. In the floor expression the numerator is divisible by z , so the apparent pole at zero is removable. $\square$

Example 7.3 (A scaled square-root series). For $q = u = 2, v = 1$, $r_j = d_j$ is the parity of j , and

$$\mu_c(j; 1/2, 2) = 2 \lfloor j/2 \rfloor, \quad H_c(z; 1/2, 2) = \frac{2z^2}{(1-z)(1-z^2)}. \quad (7.6)$$

The floor multiplicity at positive j is $2 \lfloor j/2 \rfloor + 1$. Thus their positive-level Dirichlet series differ by $\zeta(s)$ in this particular case. No such identity is assumed for other root orders.

Corollary 7.4 (Finite Hurwitz-zeta formulas). *For $\operatorname{Re} s > q$, let $C_{q,u/v}(s)$ and $F_{q,u/v}(s)$ be the Dirichlet transforms of the multiplicities at positive levels. Then*

$$C_{q,u/v}(s) = \frac{v}{u} C_q(s) - u^{-s-1} \sum_{\ell=1}^u (r_\ell - r_{\ell-1}) \zeta(s, \ell/u), \quad (7.7)$$

$$F_{q,u/v}(s) = \frac{v}{u} F_q(s) + u^{-s-1} \sum_{\ell=1}^u (d_{\ell+1} - d_\ell) \zeta(s, \ell/u), \quad (7.8)$$

where the d indices are extended periodically. These expressions give meromorphic continuation to $\mathbb{C}$.

Proof. For any period- u sequence η_j ,

$$\sum_{j \geq 1} \eta_j j^{-s} = u^{-s} \sum_{r=1}^u \eta_r \zeta(s, r/u), \quad \operatorname{Re} s > 1.$$

Partition the positive integers into residue classes and use the defining Hurwitz series. Substitute the periodic terms in Theorem 7.1. Continuation follows from that of the Hurwitz and Riemann zeta functions [5, Section 25.11]. $\square$

The periodic differences in either sum have total zero. Their potential residue at $s = 1$ therefore cancels. The pole at $s = 1$ that may remain in the full formula comes from the polynomial part, not from a nonzero mean of the periodic correction.

This gives a finite algorithm for the original scaled functions: compute one residue period and combine a rational polylogarithm with a periodic rational function. For the zeta transform, the same period produces finitely many Hurwitz values. The proof covers rational $b < 1$ as well as integer scales, and automatically retains the correct floor level zero. The possible pole locations in the geometric function need not all be actual poles, because some residues can cancel.

7.1. Scaled coefficient transforms from finitely many rotations. The finite data used above also evaluate scaled repeated-coefficient transforms of a given analytic function. Let $f(x) = \sum_{j \geq 0} a_j x^j$ have radius $R > 0$ and set $g = f - a_0$. Define $\mathcal{T}_{f,q,b}$ and $\mathcal{T}_{c,q,b}$ as in (6.1)–(6.2), replacing $n^{1/q}$ by $(bn)^{1/q}$. Put $\omega = e^{2\pi i/u}$ and

$$(\Pi_k f)(x) = \frac{1}{u} \sum_{h=0}^{u-1} \omega^{-kh} f(\omega^h x), \quad k = 0, \dots, u-1. \quad (7.9)$$

This is the usual roots-of-unity coefficient filter.

Corollary 7.5 (Derivatives and rotations). *For $|x| < R$ and $b = u/v$ in lowest terms,*

$$\mathcal{T}_{f,q,b} f = \frac{v}{u} ((\mathcal{D} + 1)^q - \mathcal{D}^q) f + \frac{1}{u} \sum_{k=0}^{u-1} (d_{k+1} - d_k) \Pi_k f, \quad (7.10)$$

$$\mathcal{T}_{c,q,b} f = \frac{v}{u} (\mathcal{D}^q - (\mathcal{D} - 1)^q) g + \frac{1}{u} \sum_{k=0}^{u-1} (r_{k-1} - r_k) \Pi_k g, \quad (7.11)$$

where the residue arrays are extended periodically.

Proof. For a monomial x^j , the multiplier of (7.9) is $u^{-1} \sum_h \omega^{h(j-k)}$, equal to one if $j \equiv k \pmod{u}$ and zero otherwise. The coefficient multipliers on the right are therefore exactly (7.2) and (7.1). Absolute convergence and the finite sums justify the identities. $\square$

For example, $q = 1, b = 2$ selects the positive even coefficients:

$$\mathcal{T}_{c,1,2}f(x) = \frac{f(x) + f(-x)}{2} - a_0. \quad (7.12)$$

These formulas are useful when f and its derivatives are accessible: the scaled transform needs derivatives of order at most $q - 1$ and u rotated evaluations. The period u need not be the smallest available period, and the formulas do not assert an optimal runtime.

8. A SCALED SPHERICAL SECTOR AND ITS PARTITION PRODUCT

We now use the rational-scale correction in a concrete model. For $q = 2, b = 2$, put $w_-(j) = \mu_c(j; 1/2, 2) = 2 \lfloor j/2 \rfloor$.

Proposition 8.1 (Odd azimuthal modes). *Let U be pullback by a rotation through π about the polar axis on S^2 , and let $\mathcal{H}_- \subset L^2(S^2)$ be its -1 eigenspace. The operator A in (4.6) preserves this subspace, and its eigenvalue j has multiplicity $w_-(j)$. Hence, for $t > 0$,*

$$\mathrm{Tr}_{\mathcal{H}_-} e^{-tA} = H_c(e^{-t}; 1/2, 2) = \frac{2e^{-2t}}{(1 - e^{-t})(1 - e^{-2t})}, \quad (8.1)$$

$$\zeta_{A,-}(s) = \zeta(s - 1) - (1 - 2^{-s})\zeta(s), \quad \mathrm{Re} s > 2. \quad (8.2)$$

Proof. The usual spherical harmonics contain the factor $e^{im\varphi}$ [5, Section 14.30]. Thus $UY_\ell^m = (-1)^m Y_\ell^m$, and the subspace selects odd m from $-\ell, \dots, \ell$. For $\ell = 2k$ their number is $2k$; for $\ell = 2k - 1$ it is also $2k$. After setting $j = \ell + 1$, both cases give $2 \lfloor j/2 \rfloor$. The rotation commutes with $\mathcal{L}$ and its spectral function A , so the restriction is well defined. The geometric trace follows from (7.6). Finally,

$$w_-(j) = j - \frac{1}{2} + \frac{1}{2}(-1)^j, \quad \sum_{j \geq 1} (-1)^j j^{-s} = (2^{1-s} - 1)\zeta(s)$$

under absolute convergence. Substitution proves (8.2). $\square$

Allowing arbitrary bosonic occupation of these modes gives

$$P_-(z) = \prod_{j \geq 1} (1 - z^j)^{-2 \lfloor j/2 \rfloor} = \mathcal{M}(z) \frac{\mathcal{P}(z^2)}{\mathcal{P}(z)}. \quad (8.3)$$

Indeed, $2 \lfloor j/2 \rfloor = j$ for even j and $j - 1$ for odd j . Divide MacMahon's product by one ordinary-partition factor at each odd level; $\prod_{\substack{j \geq 1 \\ j \text{ odd}}} (1 - z^j) = \mathcal{P}(z^2)/\mathcal{P}(z)$. This proves the equality as convergent analytic products or as formal power series. Although the displayed ratio by itself has signed coefficients, the original Euler product has nonnegative coefficients and defines the specified colored partitions. There is no mode at level one, so its coefficient of z is zero.

The standard product logarithm and Mellin formula now become

$$\log P_-(z) = \sum_{r \geq 1} \frac{H_c(z^r; 1/2, 2)}{r}, \quad (8.4)$$

$$\log P_-(e^{-t}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \zeta(s+1) \zeta_{A,-}(s) t^{-s} ds, \quad c > 2. \quad (8.5)$$

The proofs are precisely the absolute-convergence arguments used in Theorems 5.1 and 5.3, with w_- in place of c_q . Thus the scaled generating function and its Hurwitz representation are connected to the same concrete spectral sector and occupation model. The half-turn is a classical symmetry projection; the identification with these rounded counts is the worked connection.

8.1. Uniform bounds for scaled partition products. For any $q \in \mathbb{N}$ and real $b > 0$, define

$$P_{q,b}(z) = \prod_{j \geq 1} (1 - z^j)^{-\mu_c(j; 1/q, b)}. \quad (8.6)$$

Zero multiplicities are allowed. The counts have polynomial growth, so the convergence and coefficient recurrence of Section 5 apply to this product as well.

Proposition 8.2 (Uniform scaling bound). *For $0 < z < 1$ and $t > 0$,*

$$0 \leq b^{-1} H_{c,q}(z) - H_c(z; 1/q, b) \leq z, \quad (8.7)$$

$$0 \leq b^{-1} \log P_q(e^{-t}) - \log P_{q,b}(e^{-t}) \leq -\log(1 - e^{-t}). \quad (8.8)$$

In particular,

$$\log P_{q,b}(e^{-t}) \sim \frac{q! \zeta(q+1)}{b} t^{-q} \quad (t \downarrow 0). \quad (8.9)$$

Proof. Let $E_j = j^q/b - \lfloor j^q/b \rfloor$, so $0 \leq E_j < 1$. The cumulative formulas give

$$b^{-1} H_{c,q}(z) - H_c(z; 1/q, b) = (1 - z) \sum_{j \geq 1} E_j z^j.$$

This lies between zero and $(1 - z) \sum_{j \geq 1} z^j = z$. Apply this bound to $z = e^{-rt}$ and sum with weight $1/r$; absolute convergence gives (8.8) because $\sum_{r \geq 1} e^{-rt}/r = -\log(1 - e^{-t})$. This upper bound is $O(\log(1/t))$ at zero and hence $o(t^{-q})$. Theorem 5.3 proves (8.9). $\square$

For the odd spherical sector this gives $\log P_-(e^{-t}) \sim \zeta(3)t^{-2}$, half the leading logarithm for the full A spectrum. The exact residue correction still controls lower-order behavior and the low-energy spectrum. At $b = 2, z = 1/2$, the comparison (8.7) reads $0 \leq 3/2 - 4/3 = 1/6 \leq 1/2$. The integral version and decreasing-weight extension are proved in Section 13. No asymptotic formula for individual partition coefficients is asserted here.

9. INFINITE PRODUCTS, RESOLVENT SUMS AND RELATIVE DETERMINANTS

Canonical products and modified Fredholm determinants [8, 9] provide the classical normalization used here. Dowker and Kirsten [15, Sections 4, 6–7] relate zeta-value expansions, Barnes functions and subtracted spectral products. We apply these tools to the rounded spectra, derive their finite-cutoff bounds, and develop the relative and balanced products. Canonical and zeta regularization retain their respective normalizations.

9.1. Critical exponent and canonical normalization. Use only positive rounded levels. Write $d_{f,j} = \mu_f(j; a, b)$ and $d_{c,j} = \mu_c(j; a, b)$ for $j \geq 1$; any floor-zero indices are omitted. Their cumulative counts are

$$\sum_{j=1}^J d_{f,j} = \lceil (J+1)^\kappa/b \rceil - \lceil 1/b \rceil, \quad \sum_{j=1}^J d_{c,j} = \lfloor J^\kappa/b \rfloor. \quad (9.1)$$

Both are asymptotic to J^κ/b , where $\kappa = 1/a$.

Lemma 9.1 (The critical exponent). *For real σ , $\sum_{j \geq 1} d_{X,j} j^{-\sigma}$ converges if and only if $\sigma > \kappa$.*

Proof. The cumulative bound is $O(J^\kappa)$. Split the sum into dyadic intervals $[2^m, 2^{m+1})$; the contribution is $O(2^{m(\kappa-\sigma)})$, summable if $\sigma > \kappa$. Conversely, the number of levels counted with multiplicity in such an interval is asymptotic to $(2^\kappa - 1)2^{m\kappa}/b$. The contribution is bounded below by a positive constant times $2^{m(\kappa-\sigma)}$ for large m , including the case $\sigma = \kappa$. It cannot be summable then. $\square$

Let $p = \lfloor \kappa \rfloor$ and define

$$E_p(w) = (1 - w) \exp\left(\sum_{\ell=1}^p \frac{w^\ell}{\ell}\right), \quad Q_X(z) = \prod_{j \geq 1} E_p(-z/j)^{d_{X,j}}. \quad (9.2)$$

The sum is empty for $p = 0$.

Theorem 9.2 (A convergent spectral product). *The product (9.2) converges locally uniformly to an entire function. Its zeros are $-j$ with multiplicity $d_{X,j}$, and $Q_X(0) = 1$. The integer p is the least nonnegative genus for which these standard factors give a nonzero canonical product.*

On $\Omega = \{z : \operatorname{Re} z > -1\}$, use the analytic logarithm normalized at zero. Then

$$\frac{Q'_X(z)}{Q_X(z)} = (-1)^p z^p \sum_{j \geq 1} \frac{d_{X,j}}{j^p(j+z)}, \quad (9.3)$$

$$\frac{d^{p+1}}{dz^{p+1}} \log Q_X(z) = (-1)^p p! \sum_{j \geq 1} \frac{d_{X,j}}{(j+z)^{p+1}}. \quad (9.4)$$

For $|z| < 1$,

$$\log Q_X(z) = \sum_{\ell=p+1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} Z_X(\ell, 0) z^\ell, \quad Z_X(s, z) = \sum_{j \geq 1} d_{X,j} (j+z)^{-s}. \quad (9.5)$$

The original series for Z_X converges absolutely for $\operatorname{Re} s > \kappa$ on Ω .

Proof. For $|z| \leq R$ and $j > 2R$, $\log E_p(-z/j) = O_R(j^{-p-1})$, uniformly in z . Lemma 9.1 applies since $p+1 > \kappa$. The logarithmic tail therefore converges normally; finite factors give exactly the claimed zeros and no others. If a smaller genus g has $g+1 \leq \kappa$, the leading real term of $\log E_g(-z/j)$ at a fixed small positive z is $(-1)^g z^{g+1}/((g+1)j^{g+1})$. Its large- j terms have one sign, and their weighted sum diverges by the lemma. Such factors cannot define a finite nonzero product with value one at the origin.

On Ω , each $j+z$ has positive real part, so the logarithms can be chosen consistently. Differentiating the normally convergent tail gives

$$\frac{1}{j+z} + \sum_{\ell=1}^p \frac{(-1)^\ell z^{\ell-1}}{j^\ell} = \frac{(-1)^p z^p}{j^p(j+z)}.$$

This proves the first derivative. Differentiating the logarithm $p+1$ times kills all the subtraction polynomials and proves the second. To obtain (9.5), expand the same logarithm at zero. The joint series of absolute values is bounded on $|z| \leq r < 1$ by a constant times $\sum_j d_{X,j} j^{-p-1}$, so interchange is justified. Comparison with $j^{-\operatorname{Re} s}$ also proves the convergence claim for Z_X . $\square$

For a concrete operator interpretation, let L_X be diagonal on the positive-index rounded states, with eigenvalue j repeated $d_{X,j}$ times. Its inverse is compact. For an integer $m \geq 1$, membership in the Schatten class $\mathcal{S}_m$ means $\sum_j d_{X,j} j^{-m} < \infty$; thus $L_X^{-1} \in \mathcal{S}_{p+1}$. The standard eigenvalue definition of the modified Fredholm determinant [8, Equation (1.3)] gives

$$Q_X(z) = \det_{p+1}(I + zL_X^{-1}). \quad (9.6)$$

In the spherical square-root example $L_c = A$ and $L_f = B$ with its zero eigenspace removed. The derivatives in (9.4) are actual convergent traces of resolvent powers of these specified operators.

9.2. Zeta values and resolvent responses. For $a = 1/2, b = 1, p = 2$ and $d_{c,j} = 2j - 1$, so

$$Q_c(z) = \prod_{n \geq 1} \left(1 + \frac{z}{\lceil \sqrt{n} \rceil} \right) \exp \left[-\frac{z}{\lceil \sqrt{n} \rceil} + \frac{z^2}{2 \lceil \sqrt{n} \rceil^2} \right]. \quad (9.7)$$

This is the infinite product transformation (3.8) applied to normalized scalar factors. Its logarithms converge absolutely. The product-zeta bridge becomes

$$\log Q_c(z) = \sum_{\ell=3}^{\infty} \frac{(-1)^{\ell+1}}{\ell} [2\zeta(\ell-1) - \zeta(\ell)] z^\ell, \quad |z| < 1. \quad (9.8)$$

The coefficient in brackets is exactly the unscaled $C_2(\ell)$ of (2.16). Thus the zeta function gives the local coefficients of a determinant function.

For $\operatorname{Re} s > 2$ and $\operatorname{Re} z > -1$, write $j = (j+z) - z$ to get

$$Z_c(s, z) = 2\zeta(s-1, 1+z) - (2z+1)\zeta(s, 1+z). \quad (9.9)$$

In particular,

$$\frac{d^3}{dz^3} \log Q_c(z) = 4\zeta(2, 1+z) - 2(2z+1)\zeta(3, 1+z) = 2 \operatorname{Tr}(A + zI)^{-3}. \quad (9.10)$$

The displayed Hurwitz expression is derived from the absolutely convergent trace sum, not from a divergent formal resolvent trace.

The same application retains the scaled sector of Proposition 8.1: use $d_j = 2 \lfloor j/2 \rfloor$ in (9.2). Its shifted sum is

$$Z_-(s, z) = \zeta(s-1, 1+z) - z\zeta(s, 1+z) - 2^{-s}\zeta\left(s, \frac{1+z}{2}\right). \quad (9.11)$$

Indeed, $2 \lfloor j/2 \rfloor = j - \mathbf{1}_{\{j \text{ odd}\}}$; split the odd positive integers as $2k+1$. Its third logarithmic derivative is $2Z_-(3, z)$. Thus rational-scale corrections also survive in the infinite determinant response.

9.3. Finite-cutoff normalization and convergence. Let $H_M^{(r)} = \sum_{j=1}^M j^{-r}$ and $H_M = H_M^{(1)}$. For the spherical ceiling spectrum the finite normalization is

$$Q_{c,M}(z) = \frac{D_M(z)}{D_M(0)} \exp \left[-z(2M - H_M) + \frac{z^2}{2}(2H_M - H_M^{(2)}) \right]. \quad (9.12)$$

The two counterterms are finite sums over the same spectrum: $\operatorname{Tr} A_M^{-1} = 2M - H_M$ and $\operatorname{Tr} A_M^{-2} = 2H_M - H_M^{(2)}$. Taking logarithms of the finite product (4.15) proves (9.12); Theorem 9.2 then proves $Q_{c,M} \rightarrow Q_c$ locally uniformly. For $z > 0$ the unnormalized ratio $D_M(z)/D_M(0)$ itself diverges.

For $|z| \leq R$ and $M > 2R$, the logarithm of the tail product, normalized to zero at $z = 0$, satisfies

$$\left| \log \frac{Q_c(z)}{Q_{c,M}(z)} \right| \leq \frac{2R^3}{3(1 - R/(M+1))M}. \quad (9.13)$$

At zeros of the finite product the quotient means its removable tail expression. To prove the bound, expand the primary-factor logarithm: its absolute tail is at most $R^3/(3j^3(1 - R/(M+1)))$. Multiply by $2j - 1 \leq 2j$ and use $\sum_{j>M} j^{-2} \leq 1/M$. This bound gives a concrete error estimate for the limit.

9.4. Barnes evaluations and shifted zeta functions. The connection with the Barnes-zeta framework can be written directly. For $\operatorname{Re} w > 0$ and $\operatorname{Re} s > 2$, define

$$\zeta_2(s, w \mid 1, 1) = \sum_{m, n \geq 0} (w + m + n)^{-s} = \zeta(s-1, w) + (1-w)\zeta(s, w).$$

Grouping by $k = m + n$ gives $k + 1$ terms at each level. Consequently the shifted spherical sums satisfy

$$Z_c(s, z) = 2\zeta_2(s, 1+z \mid 1, 1) - \zeta(s, 1+z), \quad Z_f(s, z) = 2\zeta_2(s, 1+z \mid 1, 1) + \zeta(s, 1+z). \quad (9.14)$$

This places the evaluations within the earlier Barnes-zeta treatment [15, Sections 2–4]. The specific rounded-count realizations, relative cancellation and finite Gaussian comparison are the applications developed here.

The classical gamma and Barnes products [5, Sections 5.8 and 5.17] evaluate both spherical canonical functions:

$$Q_c(z) = G(1+z)^2 \Gamma(1+z) (2\pi)^{-z} \exp \left[(1+\gamma)z + \left(1 + \gamma - \frac{\zeta(2)}{2} \right) z^2 \right], \quad (9.15)$$

$$Q_f(z) = \frac{G(1+z)^2}{\Gamma(1+z)} (2\pi)^{-z} \exp \left[(1-\gamma)z + \left(1 + \gamma + \frac{\zeta(2)}{2} \right) z^2 \right]. \quad (9.16)$$

Here γ is Euler's constant, and Q_f uses multiplicities $2j + 1$. The expressions have the entire continuations defined by their canonical products. For a derivation, set

$$U(z) = \prod_{j \geq 1} E_2(-z/j)^j, \quad V(z) = \prod_{j \geq 1} E_2(-z/j).$$

DLMF 5.17.3 gives $U(z) = G(1+z)(2\pi)^{-z/2} e^{z/2 + (1+\gamma)z^2/2}$. The gamma product gives $V(z) = e^{-\gamma z + \zeta(2)z^2/2} / \Gamma(1+z)$. Now $Q_c = U^2/V$ and $Q_f = U^2V$, proving both constants and signs. These evaluations are useful when standard special functions are available, while (9.13) supplies a direct product-based approximation with an explicit error.

9.5. A sparse spectrum with an ordinary product. At $a = 2, b = 1$, both rounded maps have eigenvalues n^2 . They are the Dirichlet eigenvalues of $-d^2/dx^2$ on $(0, \pi)$, with eigenfunctions $\sin(nx)$. Now $\kappa = 1/2$ and $p = 0$, so the ordinary product already converges:

$$\prod_{n \geq 1} \left(1 + \frac{z}{n^2} \right) = \frac{\sinh(\pi\sqrt{z})}{\pi\sqrt{z}}. \quad (9.17)$$

This is the classical hyperbolic-sine product [5, Section 4.36], interpreted at zero by its power series. For $z > 0$, differentiation gives the convergent resolvent sum

$$\sum_{n \geq 1} \frac{1}{n^2 + z} = \frac{\pi}{2\sqrt{z}} \coth(\pi\sqrt{z}) - \frac{1}{2z}. \quad (9.18)$$

At zero its limit is $\pi^2/6$. This classical check explains the role of the critical exponent: the sparse squared spectrum needs no counterterms, whereas the square-root spherical spectrum needs two.

9.6. Floor–ceiling cancellation and relative products. For unscaled integer roots $q \geq 2$, put

$$\nu_q(j) = f_q(j) - c_q(j) = (j+1)^q - 2j^q + (j-1)^q. \quad (9.19)$$

These nonnegative weights have degree $q - 2$, with leading coefficient $q(q-1)$. Their Dirichlet series $\Delta_q(s) = \sum_{j \geq 1} \nu_q(j) j^{-s}$ converges for $\operatorname{Re} s > q - 1$ and agrees with $F_q(s) - C_q(s)$ where the individual series converge. The binomial theorem gives

$$\Delta_q(s) = \sum_{k=0}^{q-2} [1 + (-1)^{q-k}] \binom{q}{k} \zeta(s-k). \quad (9.20)$$

This is the pole-canceling difference noted in Part II [2, Remark 2]. In particular $\Delta_q(q)$ is an ordinary convergent sum, rather than subtraction of two divergent series.

Its direct product application is

$$\mathcal{R}_q(z) = \prod_{j \geq 1} E_{q-1}(-z/j)^{\nu_q(j)}. \quad (9.21)$$

It needs one fewer subtraction than the separate floor and ceiling products, because the leading multiplicities have canceled. The proof of Theorem 9.2 applies to the degree $q - 2$ weights. Denoting the separate genus- q products by $Q_{f,q}, Q_{c,q}$, cancellation of their finite factors gives

$$\frac{Q_{f,q}(z)}{Q_{c,q}(z)} = \mathcal{R}_q(z) \exp \left[\frac{(-1)^q}{q} \Delta_q(q) z^q \right] \quad (9.22)$$

on Ω , and meromorphically with the corresponding removable common zeros. Indeed, $E_q(-z/j) = E_{q-1}(-z/j) e^{(-1)^q z^q / (qj^q)}$. This gives a product form of the zeta cancellation in a relative determinant.

For the sphere, $\nu_2(j) = 2$ and the gamma product gives

$$\mathcal{R}_2(z) = \prod_{j \geq 1} \left(1 + \frac{z}{j} \right)^2 e^{-2z/j} = \frac{e^{-2\gamma z}}{\Gamma(1+z)^2}. \quad (9.23)$$

It compares the positive B spectrum with the A spectrum, whose multiplicities differ by two at every positive level. Equation (9.22) reads $Q_f/Q_c = e^{\zeta(2)z^2} \mathcal{R}_2$. The normalized relative zeta determinant is instead $\Gamma(1+z)^{-2}$, as verified below.

9.7. Balanced ordinary products and Gaussian comparisons.

Proposition 9.3 (Moment balance). *Let integer ε_ν and shifts $z_\nu \in \Omega$ satisfy $\sum_\nu \varepsilon_\nu z_\nu^\ell = 0$ for $\ell = 1, \dots, p$. Then*

$$\prod_{j \geq 1} \left[\prod_\nu (1 + z_\nu/j)^{\varepsilon_\nu} \right]^{d_{X,j}} = \prod_\nu Q_X(z_\nu)^{\varepsilon_\nu} \quad (9.24)$$

is an ordinary finite nonzero limit, absolutely convergent in logarithms.

Proof. Expand the factor logarithms for large j . Their first p coefficients vanish by moment balance, leaving $O(j^{-p-1})$. Lemma 9.1 gives absolute convergence. The same equations cancel every primary-factor exponential, which proves the identity. The finitely many remaining factors are nonzero. $\square$

For the spherical ceiling counts take $p = 2$ and shifts $0, t, 2t, 3t$ with weights $1, -3, 3, -1$, where $t > 0$. They cancel the first two moments, giving

$$\mathcal{B}(t) = \prod_{j \geq 1} \left[\frac{j(j+2t)^3}{(j+t)^3(j+3t)} \right]^{2j-1} = \frac{Q_c(2t)^3}{Q_c(t)^3 Q_c(3t)}. \quad (9.25)$$

If $\mathcal{B}_M$ denotes the product through level M , the finite Gaussian integrals of (4.18) satisfy

$$\frac{\mathcal{Z}_M(0) \mathcal{Z}_M(2t)^3}{\mathcal{Z}_M(t)^3 \mathcal{Z}_M(3t)} = \mathcal{B}_M(t)^{-1/2} \rightarrow \mathcal{B}(t)^{-1/2}. \quad (9.26)$$

The factors $(2\pi)^{M^2/2}$ cancel because the four weights also sum to zero. Thus an ordinary infinite product computes a cutoff-independent comparison of four explicitly specified Gaussian systems; no individual infinite Gaussian normalization is assumed.

The limit has a useful error bound:

$$0 \leq \log \mathcal{B}_M(t) - \log \mathcal{B}(t) \leq \frac{4t^3}{M}. \quad (9.27)$$

To prove it, apply the fundamental theorem of calculus three times to $f(x) = \log x$: $f(j+3t) - 3f(j+2t) + 3f(j+t) - f(j)$ equals the integral of $f'''(j+u+v+w) = 2/(j+u+v+w)^3$

over $[0, t]^3$. The factor logarithm in (9.25) is its negative, with modulus at most $2t^3/j^3$. Multiply by $2j - 1 \leq 2j$ and sum the tail using $\sum_{j>M} j^{-2} \leq 1/M$. In particular $0 < \mathcal{B}(t) < 1$.

The Barnes expressions give the closed evaluation

$$\mathcal{B}(t) = \frac{G(1+2t)^6}{G(1+t)^6 G(1+3t)^2} \frac{\Gamma(1+2t)^3}{\Gamma(1+t)^3 \Gamma(1+3t)}. \quad (9.28)$$

All linear and quadratic exponential normalizations cancel by the same moment equations. This supplies both a special-function evaluation and a direct finite-product approximation. For example, at $t = 1/6$ the two evaluations give

$$\mathcal{B}(1/6) \approx 0.9869734302763, \quad \lim_{M \rightarrow \infty} \frac{\mathcal{Z}_M(0) \mathcal{Z}_M(1/3)^3}{\mathcal{Z}_M(1/6)^3 \mathcal{Z}_M(1/2)} \approx 1.0065776179292.$$

If $\rho_M = \mathcal{B}_M(1/6)^{-1/2}$, the proved bound gives $\rho_M \leq \mathcal{B}(1/6)^{-1/2} \leq \rho_M e^{1/(108M)}$. At $M = 100$ this is a multiplicative error of at most $e^{1/10800} - 1$, approximately 9.26×10^{-5} . The numerical values are corroborated by the supplied program; the error guarantee follows from (9.27).

9.8. Canonical and zeta-regularized normalizations. For the sphere, (9.9) continues regularly to $s = 0$. Define

$$D_{\zeta,c}(z) = \exp[-\partial_s Z_c(0, z)]. \quad (9.29)$$

The known relationship between zeta and canonical products [9, 15] is explicit here:

$$\frac{D_{\zeta,c}(z)}{D_{\zeta,c}(0)} = Q_c(z) \exp \left[-(1+\gamma)z - \left(1 + \gamma - \frac{\zeta(2)}{2} \right) z^2 \right]. \quad (9.30)$$

We give a derivation of the normalization rather than identify two different determinants by notation. On Ω , both normalized logarithms have third derivative $2Z_c(3, z)$: differentiate $\partial_z^3 Z_c(s, z) = -s(s+1)(s+2)Z_c(s+3, z)$ at $s = 0$. The latter series is convergent near zero, so this calculation continues the identity legitimately. At $z = 0$,

$$Z_c(1+s, 0) = -s^{-1} - (1+\gamma) + O(s), \quad Z_c(2+s, 0) = 2s^{-1} + 2\gamma - \zeta(2) + O(s).$$

Using $\partial_z Z_c = -sZ_c(s+1, z)$ and $\partial_z^2 Z_c = s(s+1)Z_c(s+2, z)$ gives the first two derivatives of $\log D_{\zeta,c}$ as $-(1+\gamma)$ and $-2(1+\gamma) + \zeta(2)$. The first two derivatives of $\log Q_c$ vanish by construction. Their difference is therefore precisely the polynomial in (9.30), initially near zero and then throughout the connected half-plane.

The positive B spectrum has $Z_f(s, z) = 2\zeta(s-1, 1+z) + (1-2z)\zeta(s, 1+z)$. Hence $Z_f - Z_c = 2\zeta(s, 1+z)$, and the classical identity $\zeta'(0, w) = \log \Gamma(w) - \frac{1}{2} \log(2\pi)$ gives

$$\frac{D_{\zeta,f}(z)/D_{\zeta,c}(z)}{D_{\zeta,f}(0)/D_{\zeta,c}(0)} = \Gamma(1+z)^{-2}. \quad (9.31)$$

This verifies the claimed relative normalization.

The finite sums determine the required counterterms and observables; the finite products determine Gaussian normalizations and cutoff determinants. Their convergent infinite continuations produce resolvent traces, modified determinants and balanced Gaussian ratios. The functions F_q and C_q enter both the product coefficients and the reduction in relative-product genus. These connections use one rounded counting measure throughout the spectral and Gaussian constructions.

10. PROBABILITY AND FINITE MOMENTS

Maximum distributions and survival probabilities can be expressed through the rounded level counts. The finite power-sum formulas of [1] give their moments, while geometric inputs lead to discrete-Weibull laws.

10.1. Maxima of independent discrete uniforms.

Proposition 10.1 (Two realizations of the same law). *Let $q, M \in \mathbb{N}$. If $X_1, \dots, X_q$ are independent and uniform on $\{1, \dots, M\}$, then $J = \max_i X_i$ has distribution*

$$\mathbb{P}(J = j) = \frac{c_q(j)}{M^q}, \quad 1 \leq j \leq M. \quad (10.1)$$

The same law is obtained from $J = \lceil N^{1/q} \rceil$ when N is uniform on $\{1, \dots, M^q\}$.

Proof. Independence gives $\mathbb{P}(J \leq j) = (j/M)^q$. Taking differences proves (10.1). For the second construction, the integers n with $\lceil n^{1/q} \rceil = j$ satisfy $(j-1)^q < n \leq j^q$, so there are exactly $c_q(j)$ of them. $\square$

For an integer $p \geq 0$, the finite ceiling power sums therefore calculate

$$\mathbb{E}[J^p] = M^{-q} \sum_{j=1}^M c_q(j) j^p = M^{-q} \sum_{k=0}^{q-1} (-1)^{q-k+1} \binom{q}{k} \sum_{j=1}^M j^{p+k}. \quad (10.2)$$

The ordinary power sums use the classical formulas, as in Part I, Section 4 [1]. For $q = 2, p = 1$,

$$\mathbb{E}[J] = M^{-2} \left(2 \sum_{j=1}^M j^2 - \sum_{j=1}^M j \right) = \frac{(M+1)(4M-1)}{6M}, \quad (10.3)$$

$$\mathbb{E}[J^2] = \frac{(M+1)(3M^2 + M - 1)}{6M}. \quad (10.4)$$

At $M = 10$ the mean is $143/20 = 7.15$.

The cubical shell and ceiling block have the same count $c_q(j)$. For independent uniform task durations, their maximum is the total completion time, with moments given by (10.2).

10.2. Geometric inputs and discrete survival laws. Let N have the positive geometric law $\mathbb{P}(N = n) = (1-r)r^{n-1}$, $n \geq 1$, with $0 < r < 1$. Its tail is $\mathbb{P}(N > k) = r^k$ for integers $k \geq 0$.

Theorem 10.2 (Rounded geometric survival law). *For $a, b > 0$ and $J = \lceil (bN)^a \rceil$,*

$$\mathbb{P}(J > j) = r^{\lceil j^{1/a}/b \rceil}, \quad j \in \mathbb{N}_0. \quad (10.5)$$

For $a = 1/q, b = 1$ this specializes to

$$\mathbb{P}(J > j) = r^{j^q}, \quad (10.6)$$

$$\mathbb{P}(J = j) = r^{(j-1)^q} - r^{j^q}, \quad j \geq 1. \quad (10.7)$$

In this specialization, for $p \in \mathbb{N}$,

$$\mathbb{E}[J^p] = \sum_{j \geq 0} ((j+1)^p - j^p) r^{j^q}. \quad (10.8)$$

Proof. For integer j , $\lceil x \rceil > j$ is equivalent to $x > j$. Thus $J > j$ if and only if $N > j^{1/a}/b$, which is the same as $N > \lceil j^{1/a}/b \rceil$. This proves the general tail, and taking its successive differences proves the mass function. For a positive integer-valued J ,

$$J^p = \sum_{j=0}^{J-1} ((j+1)^p - j^p).$$

Tonelli's theorem gives the moment formula. Its summands are a polynomial times r^{j^q} , so every such moment is finite. $\square$

In the convention supported on $\mathbb{N}_0$, $J-1$ has the discrete-Weibull mass function $r^{k^q} - r^{(k+1)^q}$. The distributional family is established in the discrete reliability literature [10]; the rounded geometric construction above is derived directly from inverse level thresholds.

Corollary 10.3 (A theta-function mean). *For $q = 2$,*

$$\mathbb{E}[J] = \sum_{j \geq 0} r^{j^2} = \frac{1 + \vartheta_3(0, r)}{2}. \quad (10.9)$$

At $r = 1/2$, $\mathbb{E}[J] \simeq 1.56446841360594$.

Proof. Use (10.8) at $p = 1$ and the normalization $\vartheta_3(0, r) = 1 + 2 \sum_{j \geq 1} r^{j^2}$ [5, Equation 20.2.3]. $\square$

Other input laws are treated by evaluating their survival functions at the same inverse thresholds.

11. NATURAL BOUNDARIES AT IRRATIONAL LINEAR SCALES

At linear scales these functions belong to the Beatty and step-sequence generating-function setting [7]. The Pólya–Carlson theorem [11] gives a natural boundary when the integer coefficients have radius one and are not eventually periodic.

A different analytic behavior occurs for $a = 1$ and irrational $b = \alpha > 1$.

Theorem 11.1 (Natural boundary). *For every irrational $\alpha > 1$, both $H_c(z; 1, \alpha)$ and $H_f^{(+)}(z; 1, \alpha)$ have radius of convergence one and the unit circle as a natural boundary.*

Proof. The ceiling coefficients are

$$w_j = \lfloor j/\alpha \rfloor - \lfloor (j-1)/\alpha \rfloor \in \{0, 1\}, \quad \sum_{j=1}^N w_j = \lfloor N/\alpha \rfloor.$$

Thus there are infinitely many ones and their density is $1/\alpha$, an irrational number. The coefficients of the positive-index floor series are also in $\{0, 1\}$, and their cumulative count up to N is $\lceil (N+1)/\alpha \rceil - 1$. Their density is the same. In either case the coefficients are bounded by one and equal one infinitely often, so Cauchy–Hadamard gives radius one.

Neither coefficient sequence can be eventually periodic, since an eventually periodic binary sequence has rational density. If its generating function were rational, its coefficients would, after finitely many indices, satisfy a constant-coefficient linear recurrence. The recurrence determines the next coefficient from a fixed number of preceding binary coefficients. There are finitely many such states. Along the infinite sequence a state must recur, forcing eventual periodicity, which is impossible.

Apply the classical Pólya–Carlson theorem: for an integer-coefficient series with convergence radius one, failure to be a rational function forces the entire unit circle to obstruct analytic continuation [11, Section 1]. This proves the assertion. $\square$

A *natural boundary* obstructs analytic continuation through every point of the boundary circle.

The count supplies the classical theorem’s hypotheses. At rational linear scales the coefficients are periodic and the corresponding functions are rational. The nonlinear case requires separate analysis.

12. BEATTY SEQUENCES AND BALANCED ALLOCATION

The inverse counts give a generating-function form of Beatty complementation [7] and a mechanical-word allocation rule [12].

12.1. Complementary floor levels.

Proposition 12.1 (A generating-function form of Beatty complementation). *Let $\alpha > 1$ be irrational and $\beta = \alpha/(\alpha - 1)$. Then for $|z| < 1$,*

$$H_f^{(+)}(z; 1, \alpha) + H_f^{(+)}(z; 1, \beta) = \frac{z}{1 - z}. \quad (12.1)$$

Equivalently, the positive integers $\lfloor \alpha n \rfloor$ and $\lfloor \beta n \rfloor$, $n \geq 1$, form disjoint complementary sequences.

Proof. The reciprocals satisfy $1/\alpha + 1/\beta = 1$. For $j \geq 1$, the two noninteger numbers j/α and j/β sum to j , so their ceilings sum to $j + 1$. Subtract this identity at j from its version at $j + 1$. The result is $\mu_f(j; 1, \alpha) + \mu_f(j; 1, \beta) = 1$. Both positive-index series have no level zero because $\alpha, \beta > 1$. Each positive coefficient therefore equals one in the sum of their generating functions. $\square$

For $\alpha = \phi = (1 + \sqrt{5})/2$, $\beta = \phi^2$, the sequences begin

$$1, 3, 4, 6, 8, 9, 11, \dots \quad \text{and} \quad 2, 5, 7, 10, 13, 15, 18, \dots$$

The multiplicity formula thus yields complementary Beatty sequences and connects their generating functions with balanced words [12].

12.2. A deterministic allocation rule.

Proposition 12.2 (Balanced allocation). *Let $0 < \rho < 1$ and define*

$$u_j = \lfloor (j + 1)\rho \rfloor - \lfloor j\rho \rfloor, \quad j \geq 0. \quad (12.2)$$

Then $u_j \in \{0, 1\}$, and

$$\sum_{j=0}^{N-1} u_j = \lfloor N\rho \rfloor, \quad \left| \sum_{j=0}^{N-1} u_j - N\rho \right| < 1. \quad (12.3)$$

The number of ones in any block of length L is either $\lfloor L\rho \rfloor$ or $\lceil L\rho \rceil$. In particular, any two such blocks differ by at most one.

Proof. The increment ρ is between zero and one, proving the binary property. Telescope for the prefix formula. For a block starting at m the count is $\lfloor (m + L)\rho \rfloor - \lfloor m\rho \rfloor = \lfloor L\rho + \{m\rho\} \rfloor$, which has one of the two claimed values. This also proves the block balance. $\square$

Assign slot j to resource A when $u_j = 1$ and to B otherwise. The prefix discrepancy is less than one slot, and equal-length blocks differ in their A allocations by at most one. For rational ρ the pattern is periodic. For irrational ρ it is an aperiodic mechanical word: eventual periodicity would force a rational limiting density, contrary to (12.3).

With $b = 1/\rho$, these are linear ceiling multiplicities after an index shift. The count therefore gives both a generating function and a two-resource schedule with the proved discrepancy bound.

13. INTEGRAL COUNTERPARTS AND AN ERROR BOUND

Replacing a discrete fiber count by its interval length yields the integral counterpart suggested in [2, Section 11.4]. Summation by parts bounds the discrete–continuous difference.

13.1. Replacing counts by interval lengths.

Theorem 13.1 (Integral level formulas). *Let $a, b > 0$ and $\kappa = 1/a$. For a nonnegative weight w , or under absolute integrability,*

$$\int_0^\infty w(\lceil (bx)^a \rceil) dx = \frac{1}{b} \sum_{j \geq 1} (j^\kappa - (j-1)^\kappa) w(j), \quad (13.1)$$

$$\int_0^\infty w(\lfloor (bx)^a \rfloor) dx = \frac{1}{b} \sum_{j \geq 0} ((j+1)^\kappa - j^\kappa) w(j). \quad (13.2)$$

Proof. The ceiling value is j on $((j-1)^\kappa/b, j^\kappa/b]$, whose length is $(j^\kappa - (j-1)^\kappa)/b$. The single point $x = 0$ has measure zero. The floor value is j on $[j^\kappa/b, (j+1)^\kappa/b)$, with the other displayed length. Integrate over these disjoint intervals and use Tonelli's theorem or absolute integrability. $\square$

The integral construction proposed in Part II, Section 11.4 [2], thus replaces a discrete count by interval length. For $\kappa = q \in \mathbb{N}$, the weights are $c_q(j)/b$ and $f_q(j)/b$.

13.2. A bound uniform in the number of levels.

Theorem 13.2 (Discrete–continuous comparison). *Suppose $w(j) \geq 0$ decreases to zero on $\mathbb{N}$ and the integral $I = \int_0^\infty w(\lceil (bx)^a \rceil) dx$ is finite. Then $S = \sum_{n \geq 1} w(\lceil (bn)^a \rceil)$ is finite and*

$$0 \leq I - S \leq w(1). \quad (13.3)$$

Proof. Write $V_j = j^\kappa/b$, $N_j = \lfloor V_j \rfloor$ and $E_j = V_j - N_j \in [0, 1)$, with all three zero at $j = 0$. For a finite number of levels M , summation by parts gives

$$I_M - S_M = E_M w(M) + \sum_{j=1}^{M-1} E_j (w(j) - w(j+1)), \quad (13.4)$$

where $I_M = \sum_{j=1}^M (V_j - V_{j-1}) w(j)$ and $S_M = \sum_{j=1}^M (N_j - N_{j-1}) w(j)$. All terms on the right are nonnegative. Since $0 \leq E_j < 1$, their sum is at most $w(M) + \sum_{j=1}^{M-1} (w(j) - w(j+1)) = w(1)$. Thus $0 \leq S_M \leq I_M \leq I$, proving convergence of S_M . Passing to the limit proves the bound; $E_M w(M) \rightarrow 0$. $\square$

For $a = 1/q$ and $w(j) = z^j$, $0 < z < 1$, Theorem 13.1 and (13.3) give

$$0 \leq \frac{H_{c,q}(z)}{b} - H_c(z; 1/q, b) \leq z. \quad (13.5)$$

This recovers (8.7) and its $1/6$ gap in the $q = b = 2, z = 1/2$ example of Section 8.

The cumulative count differs from interval length by less than one. Summation by parts transfers this discrepancy to decreasing weights; for oscillatory weights, an estimate in discrete total variation can replace monotonicity.

14. RECIPROCAL-FIBONACCI MOMENTS AND GENERATING FUNCTIONS

Reciprocal-Fibonacci sums admit Lambert-series representations [13, 14]. The moment family defined in [2] can be organized into one analytic generator, giving a rational-term expansion and an explicit tail bound.

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Part II, Section 6, equation (13), defines the Shah–Pingala function

$$S(q) = \sum_{n \geq 1} \frac{n^q}{F_n}, \quad q \in \mathbb{N}_0. \quad (14.1)$$

Here F_n denotes a Fibonacci number; the function $F_q(s)$ in (2.14) is the floor Dirichlet transform.

The following generator also proves convergence of every $S(q)$.

Proposition 14.1 (Moment generator). *The power series*

$$G_F(z) = \sum_{n \geq 1} \frac{z^n}{F_n} \quad (14.2)$$

has radius of convergence $\phi = (1 + \sqrt{5})/2$. For $q \in \mathbb{N}_0$,

$$S(q) = (z \, d/dz)^q G_F(z)|_{z=1} = \frac{d^q}{dt^q} G_F(e^t) \Big|_{t=0}. \quad (14.3)$$

Proof. Binet's formula gives $F_n^{1/n} \rightarrow \phi$, so Cauchy–Hadamard gives radius ϕ . The point $z = 1$ is strictly inside that disk, and termwise differentiation multiplies the n th coefficient by n^q . The substitution $z = e^t$ changes $z \, d/dz$ to d/dt . $\square$

Set $r_k = (-1)^k \phi^{-(2k+1)}$, $k \geq 0$.

Theorem 14.2 (Rational-term expansion). *For $|z| < \phi$ and $q \in \mathbb{N}_0$,*

$$G_F(z) = \sqrt{5} \sum_{k \geq 0} \frac{r_k z}{1 - r_k z}, \quad (14.4)$$

$$S(q) = \sqrt{5} \sum_{k \geq 0} \text{Li}_{-q}(r_k). \quad (14.5)$$

For a truncation at $k = K$, the latter tail satisfies

$$\left| \sqrt{5} \sum_{k \geq K} \text{Li}_{-q}(r_k) \right| \leq \sqrt{5} \frac{\text{Li}_{-q}(\phi^{-1})}{\phi^{-1}} \frac{\phi^{-(2K+1)}}{1 - \phi^{-2}}. \quad (14.6)$$

Proof. By Binet's formula,

$$\frac{1}{F_n} = \frac{\sqrt{5} \phi^{-n}}{1 - (-1)^n \phi^{-2n}} = \sqrt{5} \sum_{k \geq 0} ((-1)^k \phi^{-(2k+1)})^n.$$

To interchange the sums after multiplication by z^n , observe that

$$\sum_{n \geq 1} \sum_{k \geq 0} |z|^n |r_k|^n = \sum_{n \geq 1} \frac{(|z|/\phi)^n}{1 - \phi^{-2n}} \leq \frac{1}{1 - \phi^{-2}} \sum_{n \geq 1} (|z|/\phi)^n < \infty.$$

Summing the inner geometric series gives (14.4). Multiplication by n^q preserves absolute convergence at $z = 1$, proving (14.5).

Let $\rho = \phi^{-1}$. For $|r| \leq \rho$,

$$|\text{Li}_{-q}(r)| \leq \sum_{n \geq 1} n^q |r|^n \leq |r| \sum_{n \geq 1} n^q \rho^{n-1} = |r| \frac{\text{Li}_{-q}(\rho)}{\rho}.$$

Finally, $\sum_{k \geq K} |r_k| = \phi^{-(2K+1)}/(1 - \phi^{-2})$. $\square$

The Binet/geometric expansion is the classical Lambert-series method [13, 14]; Euler differentiation gives the full moment family from one generator.

Every summand in (14.5) is a rational function of r_k , since $\text{Li}_{-q}(r) = (r \, d/dr)^q [r/(1 - r)]$. For example,

$$\text{Li}_0(r) = \frac{r}{1 - r}, \quad \text{Li}_{-1}(r) = \frac{r}{(1 - r)^2}, \quad \text{Li}_{-2}(r) = \frac{r(1 + r)}{(1 - r)^3}.$$

The first values are

$$\begin{aligned} S(0) &\simeq 3.35988566624318, \\ S(1) &\simeq 9.32045171228187, \\ S(2) &\simeq 40.1589010753011. \end{aligned}$$

Normalizing the reciprocal-Fibonacci weights gives

$$\mathbb{P}(N = n) = \frac{1}{S(0)F_n}, \quad n \geq 1. \quad (14.7)$$

They are positive and sum to one. The moments and moment generating function are

$$\mathbb{E}[N^q] = \frac{S(q)}{S(0)}, \quad M_N(t) = \frac{G_F(e^t)}{S(0)}, \quad t < \log \phi. \quad (14.8)$$

The restriction on t is exact: for $t \geq \log \phi$ the summands e^{tn}/F_n do not tend to zero. Numerically the mean and variance are 2.77403835669904 and 4.25717180420114, respectively.

Thus one coefficient transform organizes the inherited moment family and gives a computable expansion with the bound (14.6).

15. FINITE COMPRESSION AND POWER-BOUNDED DOUBLE SUMS

The finite level formulas [1] reduce the number of weight evaluations by grouping complete fibers and retaining the last partial block.

15.1. Single-index block formulas.

Theorem 15.1 (Finite block formulas). *Let $q, N \in \mathbb{N}$ and let w be any weight on the positive integers. Put $m = \lfloor N^{1/q} \rfloor$ and $M = \lceil N^{1/q} \rceil$. Then*

$$\sum_{n=1}^N w(\lfloor n^{1/q} \rfloor) = \sum_{j=1}^{m-1} f_q(j)w(j) + (N - m^q + 1)w(m), \quad (15.1)$$

$$\sum_{n=1}^N w(\lceil n^{1/q} \rceil) = \sum_{j=1}^{M-1} c_q(j)w(j) + (N - (M-1)^q)w(M). \quad (15.2)$$

Proof. For the floor, complete positive blocks are $j^q, \dots, (j+1)^q - 1$ for $1 \leq j < m$. The last block is $m^q, \dots, N$, with length $N - m^q + 1$. For the ceiling, complete blocks are $(j-1)^q + 1, \dots, j^q$ for $1 \leq j < M$. The last is $(M-1)^q + 1, \dots, N$. Summing the constant weight on each block gives the formulas. They include the case of a perfect power N without changing any endpoint. $\square$

The original repetition method [1] uses m or M weight evaluations. For fixed $q > 1$ this is $O(N^{1/q})$; at $q = 2, N = 10^6$, both formulas require 1,000 evaluations. An integer root and the polynomial block sizes determine the ranges.

This reduction is useful when weight evaluation is costly. The arithmetic overhead and available closed forms still determine runtime; for $a > 1$, rounded levels need not repeat frequently.

15.2. Compression of power-bounded double sums. The weighted finite identity of Proposition 3.2 also applies when the input set consists of integer pairs. We use a power-bounded region to connect finite compression with a theta trace, a fractional zeta pole and a quantitative occupation-product expansion. The reordering is elementary counting by columns; the theta transformation used below is classical [5, Section 20.7].

For $\rho > 0$ and $n \in \mathbb{N}$, put

$$\mathcal{R}_{\rho,n} = \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq n, \quad i \leq j \leq \lfloor i^\rho \rfloor\}, \quad S_{\rho,n}(w) = \sum_{(i,j) \in \mathcal{R}_{\rho,n}} w(j). \quad (15.3)$$

An empty inner interval contributes zero. If $\rho = p/(q+1)$, where $p, q > 0$, the upper condition is equivalently $j \leq i^p/j^q$. This is a comparison with the running index j ; the different condition $i^p/j^q > 1$ has a different cutoff.

Theorem 15.2 (Column counts and double-sum compression). *For $\rho > 0$, let $M = \lfloor n^\rho \rfloor$ and $[x]_+ = \max(x, 0)$. Then*

$$d_{\rho,n}(j) = [\min(n, j) - \lceil j^{1/\rho} \rceil + 1]_+, \quad 1 \leq j \leq M, \quad (15.4)$$

$$S_{\rho,n}(w) = \sum_{j=1}^M d_{\rho,n}(j)w(j). \quad (15.5)$$

In particular, for $\rho = p/(q+1)$ this uses $O(\lfloor n^{p/(q+1)} \rfloor)$ weight and root evaluations.

Proof. For fixed j , the admissible first indices satisfy $\lceil j^{1/\rho} \rceil \leq i \leq \min(n, j)$. Counting this interval proves (15.4); summing its identical weights proves (15.5). $\square$

The complexity statement counts evaluations, with fixed parameters; it does not assert a bit-complexity bound. For $\rho > 1$, direct enumeration visits $\Theta(n^{\rho+1})$ pairs, whereas (15.5) has $\lfloor n^\rho \rfloor$ terms. If a prefix $W(m) = \sum_{j=1}^m w(j)$ can be evaluated directly, the alternative $S_{\rho,n}(w) = \sum_{i=1}^n [W(\lfloor i^\rho \rfloor) - W(i-1)]$ uses only n prefix differences. At $\rho = 1$ the region is the diagonal; for $0 < \rho < 1$ it consists solely of $(1, 1)$. The remaining applications therefore assume $\rho > 1$.

Let $\beta = 1/\rho \in (0, 1)$ and let $\mathcal{R}_{\rho,\infty} = \bigcup_n \mathcal{R}_{\rho,n}$. Its column counts are

$$d_\rho(j) = j - \lceil j^\beta \rceil + 1, \quad 1 \leq d_\rho(j) \leq j, \quad d_{\rho,n}(j) \longrightarrow d_\rho(j). \quad (15.6)$$

Thus the infinite regrouping is valid whenever $\sum_{j \geq 1} j|w(j)| < \infty$, or for nonnegative weights with an extended value. On $\ell^2(\mathcal{R}_{\rho,\infty})$, define $L_\rho e_{i,j} = j e_{i,j}$ with its natural diagonal domain. This positive self-adjoint model has compact inverse and eigenvalue j of multiplicity $d_\rho(j)$.

15.3. A quadratic cutoff and its theta trace. For $\rho = 2$, write $H_n(z) = S_{2,n}(j \mapsto z^j)$, $H(z) = \sum_{j \geq 1} d_2(j)z^j$, and $\Theta_n(z) = \sum_{k=1}^n z^{k^2}$.

Theorem 15.3 (Theta evaluation and finite-cutoff error). *For $|z| < 1$,*

$$H_n(z) = \frac{z(1-z^n)}{(1-z)^2} - \frac{z}{1-z} \Theta_n(z), \quad (15.7)$$

$$H(z) = \frac{z}{(1-z)^2} - \frac{z}{1-z} \sum_{k \geq 1} z^{k^2}. \quad (15.8)$$

For $t > 0$, put $K_n(t) = H_n(e^{-t})$ and $K(t) = H(e^{-t}) = \text{Tr } e^{-tL_2}$. Then

$$0 \leq K(t) - K_n(t) \leq \frac{e^{-t(n+1)}}{(1-e^{-t})^2}. \quad (15.9)$$

As $t \downarrow 0$,

$$K(t) = t^{-2} - \frac{\sqrt{\pi}}{2} t^{-3/2} + \frac{1}{2} t^{-1} + O(t^{-1/2}). \quad (15.10)$$

Proof. The row sum is $\sum_{j=i}^{i^2} z^j = (z^i - z^{i^2+1})/(1-z)$. Summing over $i \leq n$ proves (15.7). The bound $d_2(j) \leq j$ justifies the infinite limit and proves (15.8). For a fixed $j > n$, at most $j - n$ states have been omitted, and none are omitted at $j \leq n$. Therefore $0 \leq K - K_n \leq \sum_{j > n} (j - n)e^{-tj}$, giving (15.9). The same argument bounds $|H(z) - H_n(z)|$ by $r^{n+1}/(1-r)^2$ on $|z| \leq r < 1$.

The Jacobi transformation [5, Equation 20.7.32] gives

$$\sum_{k \geq 1} e^{-tk^2} = \frac{\sqrt{\pi}}{2\sqrt{t}} - \frac{1}{2} + O(t^{-1/2}e^{-\pi^2/t}).$$

Substitute this into (15.8) and use $e^{-t}/(1-e^{-t})^2 = t^{-2} + O(1)$ and $(e^t - 1)^{-1} = t^{-1} - 1/2 + O(t)$. Keeping terms through t^{-1} proves (15.10). $\square$

The fractional power measures the deficit $j - d_2(j) = \lceil \sqrt{j} \rceil - 1$ from the multiplicity j . It is a feature of the specified diagonal model; its finite approximations have the explicit error (15.9).

15.4. A fractional pole in the zeta transform.

Theorem 15.4 (Meromorphic continuation to a half-plane). *For $\rho > 1$, $\beta = 1/\rho$ and $\operatorname{Re} s > 2$, define*

$$Z_\rho(s) = \operatorname{Tr} L_\rho^{-s} = \sum_{j \geq 1} d_\rho(j) j^{-s}.$$

Then

$$Z_\rho(s) = \zeta(s-1) - \zeta(s-\beta) + E_\rho(s), \quad E_\rho(s) = \sum_{j \geq 1} (j^\beta - \lceil j^\beta \rceil + 1) j^{-s}. \quad (15.11)$$

The function E_ρ is holomorphic on $\operatorname{Re} s > 1$. Consequently Z_ρ continues meromorphically to that half-plane, with precisely two simple poles: residue 1 at $s = 2$ and residue -1 at $s = 1 + \beta$.

Proof. Write $d_\rho(j) = j - j^\beta + \varepsilon_j$, where $0 < \varepsilon_j \leq 1$. Absolute convergence gives (15.11) on $\operatorname{Re} s > 2$. The bound on ε_j gives locally uniform convergence of E_ρ on $\operatorname{Re} s > 1$. The two shifted zeta poles are distinct and cannot be canceled by this holomorphic remainder. $\square$

For $\rho = 2$ the theta trace gives a further continuation. Mellin integration initially yields $\Gamma(s)Z_2(s) = \int_0^\infty t^{s-1} K(t) dt$ for $\operatorname{Re} s > 2$. Subtracting the three terms of (15.10) on $(0, 1)$ gives

$$\Gamma(s)Z_2(s) = \frac{1}{s-2} - \frac{\sqrt{\pi}/2}{s-3/2} + \frac{1/2}{s-1} + F(s), \quad \operatorname{Re} s > \frac{1}{2}, \quad (15.12)$$

where F is holomorphic there. The integral over $(1, \infty)$ is entire because $K(t)$ decays exponentially. Thus Z_2 has poles at $2, 3/2, 1$ with respective residues $1, -1, 1/2$ in this larger half-plane. In particular, the $t^{-3/2}$ term and the pole at $3/2$ describe the same counting correction through different transforms.

15.5. A fractional correction to an occupation product. Use the multiplicities (15.6) in the occupation construction of Section 5:

$$\mathcal{P}_\rho(z) = \prod_{j \geq 1} (1 - z^j)^{-d_\rho(j)}, \quad \mathcal{P}_{\rho,n}(z) = \prod_{j=1}^{\lfloor n^\rho \rfloor} (1 - z^j)^{-d_{\rho,n}(j)}. \quad (15.13)$$

On $|z| < 1$ the logarithms converge locally uniformly, since $d_{\rho,n}(j) \leq d_\rho(j) \leq j$ and $\sum_{j \geq 1} j r^j < \infty$ for $r < 1$. Dominated convergence proves $\mathcal{P}_{\rho,n} \rightarrow \mathcal{P}_\rho$ locally uniformly.

Theorem 15.5 (Partition logarithm and cutoff control). *For $t > 0$,*

$$\log \mathcal{P}_2(e^{-t}) = \sum_{r \geq 1} \frac{K(rt)}{r}. \quad (15.14)$$

As $t \downarrow 0$,

$$\log \mathcal{P}_2(e^{-t}) = \zeta(3)t^{-2} - \frac{\sqrt{\pi}}{2}\zeta(5/2)t^{-3/2} + \frac{1}{2}\zeta(2)t^{-1} + O(t^{-1/2}). \quad (15.15)$$

The finite products satisfy

$$0 \leq \log \frac{\mathcal{P}_2(e^{-t})}{\mathcal{P}_{2,n}(e^{-t})} \leq \frac{-\log(1 - e^{-(n+1)t})}{(1 - e^{-t})^2}. \quad (15.16)$$

Proof. Expand $-\log(1-z^j) = \sum_{r \geq 1} z^{jr}/r$. Absolute convergence, or Tonelli for real $z \in (0, 1)$, proves (15.14). The remainder in (15.10) is bounded by $Ct^{-1/2}$ for all $t > 0$: near zero this is the theorem, and on $t \geq 1$ it follows from continuity, exponential decay of K , and decay of the three subtracted powers. Hence its contribution to (15.14) is at most $Ct^{-1/2} \sum_{r \geq 1} r^{-3/2}$. Summing the three explicit powers proves (15.15). Apply (15.9) at rt and use $1 - e^{-rt} \geq 1 - e^{-t}$ to obtain

$$0 \leq \log \frac{\mathcal{P}_2(e^{-t})}{\mathcal{P}_{2,n}(e^{-t})} \leq \frac{1}{(1 - e^{-t})^2} \sum_{r \geq 1} \frac{e^{-rt(n+1)}}{r},$$

which is (15.16). $\square$

The Mellin representation of Theorem 5.3 likewise becomes

$$\log \mathcal{P}_\rho(e^{-t}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \zeta(s+1) Z_\rho(s) t^{-s} ds, \quad c > 2.$$

Absolute convergence on this line justifies the formula. For $\rho = 2$, the fractional pole supplies the explicit secondary term in (15.15). This is an asymptotic expansion of the product logarithm. Asymptotics of its individual coefficients require further coefficient-extraction hypotheses and analysis; general multiple-pole results are developed in [4, 16]. The finite compression, theta evaluation, zeta continuation and occupation expansion are four applications of the same column counts.

16. EVALUATION AND COMPUTATIONAL VERIFICATION

The same count supplies the spectral, Gaussian and occupation quantities considered here. For $c_2(j) = 2j - 1$, the geometric transform is the trace of e^{-tA} , the Dirichlet transform is $\zeta_A(s)$, and the Euler product counts unrestricted occupations of its modes. The finite product gives a cutoff determinant and a Gaussian normalization; its convergent normalization gives a determinant function with resolvent derivatives. The counts $\mu_c(j; 1/2, 2) = 2 \lfloor j/2 \rfloor$ produce the corresponding quantities on the odd azimuthal sector.

Polynomial counts reduce analytic coefficient transforms to finitely many derivatives. Positive rational scales add a finite residue period, which supplies rational generating functions, Hurwitz-zeta expressions and the finite rotations of (7.9). These representations are useful when derivatives, standard special functions or symmetry restrictions can be evaluated directly. Applying the same construction to another geometry requires its eigenvalues and multiplicities to be specified first.

For power-bounded pairs, the same counts determine a theta trace, a fractional zeta pole and a secondary partition term. Equations (15.9) and (15.16) control the finite approximations.

Computational checks. The source package includes the original verification programs `anc/verify_applications.py`, `anc/verify_revision.py` and `anc/verify_sum_products.py`, together with `anc/verify_power_bounded.py` for Section 15.2. They require Python 3.10 or later and use the standard library only. From the package root, run the combined driver `python3 anc/verify_all.py > rerun_results.json`

Its recorded output is `anc/all_results.json`. The inherited result files and the new record `anc/power_bounded_results.json` are also included. The execution instructions, numerical settings and explicit tolerances are documented in `anc/README.md`.

Exact checks compare directly enumerated counts with clipped, weighted and residue formulas, and compare independently computed partition coefficients. They also verify cutoff determinant and Gaussian identities, counterterms and relative multiplicity polynomials. Numerical checks evaluate finite rotations, scaled traces, canonical and balanced-product tails, probability moments, the theta mean, reciprocal-Fibonacci evaluations, Beatty allocation and integral comparisons. Gamma and Barnes evaluations use separate asymptotic/recurrence calculations and are compared with zeta-series evaluations. Integer and rational arithmetic

gives exact finite identities; Decimal and floating-point comparisons provide numerical corroboration. The convergence and continuation results follow from the proofs in the text.

Additional checks compare power-bounded row and column counts at exact endpoints, finite theta expressions, partition coefficients and both cutoff bounds. Numerical evaluations corroborate the asymptotic coefficients; the proofs in Section 15.4 establish analytic continuation.

REFERENCES

- [1] D. Shah, M. Sahni, R. Sahni, E. León-Castro and M. Olazabal-Lugo, *Series of Floor and Ceiling Function—Part I: Partial Summations*, Mathematics **10**(7) (2022), article 1178. doi:10.3390/math10071178.
- [2] D. Shah, M. Sahni, R. Sahni, E. León-Castro and M. Olazabal-Lugo, *Series of Floor and Ceiling Functions—Part II: Infinite Series*, Mathematics **10**(9) (2022), article 1566. doi:10.3390/math10091566.
- [3] M. Lachièze-Rey, *A new basis for eigenmodes on the Sphere*, arXiv:math/0304409v2 (2003), introduction and Section 2.
- [4] B. L. Granovsky, D. Stark and M. Erlihson, *Meinardus' theorem on weighted partitions: extensions and a probabilistic proof*, Advances in Applied Mathematics **41**(3) (2008), 307–328. doi:10.1016/j.aam.2007.11.001.
- [5] NIST Digital Library of Mathematical Functions, <https://dlmf.nist.gov/>, Sections 2.5 (Mellin transforms), 4.36 (hyperbolic products), 5.8 (gamma products), 5.17 (Barnes functions), 5.11 (gamma asymptotics), 14.30 (spherical harmonics), 20.2 and 20.7 (theta functions and transformations), 25.11 (Hurwitz zeta), and 25.12 (polylogarithms). Accessed 12–15 October 2026.
- [6] M. Beck and S. Robins, *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra*, second edition, Springer, 2015. doi:10.1007/978-1-4939-2969-6.
- [7] K. O'Bryant, *A generating function technique for Beatty sequences and other step sequences*, Journal of Number Theory **94** (2002), 299–319.
- [8] T. Britz, A. Carey, F. Gesztesy, R. Nichols, F. Sukochev and D. Zanin, *The product formula for regularized Fredholm determinants*, Proceedings of the American Mathematical Society, Series B **8** (2021), 42–51. doi:10.1090/bproc/70. Preprint: arXiv:2007.12834v3.
- [9] J. R. Quine, S. H. Heydari and R. Y. Song, *Zeta regularized products*, Transactions of the American Mathematical Society **338**(1) (1993), 213–231.
- [10] R. Vila, E. Y. Nakano and H. Saulo, *Theoretical results on the discrete Weibull distribution of Nakagawa and Osaki*, Statistics **53**(2) (2019), 339–363. doi:10.1080/02331888.2018.1550645.
- [11] J. Bell, R. Miles and T. Ward, *Towards a Pólya–Carlson dichotomy for algebraic dynamics*, Indagationes Mathematicae **25**(4) (2014), 652–668. doi:10.1016/j.indag.2014.04.005.
- [12] G. Paquin and L. Vuillon, *A characterization of balanced episturmian sequences*, The Electronic Journal of Combinatorics **14**(1) (2007), research paper R33. doi:10.37236/951.
- [13] R. André-Jeannin, *Lambert series and the summation of reciprocals in certain Fibonacci–Lucas-type sequences*, Fibonacci Quarterly **28**(3) (1990), 223–226. <https://www.fq.math.ca/Scanned/28-3/andre-jeannin.pdf>.
- [14] R. S. Melham, *Lambert series and elliptic functions and certain reciprocal sums*, Fibonacci Quarterly **37**(3) (1999), 208–212. <https://www.mathstat.dal.ca/FQ/Scanned/37-3/melham1.pdf>.
- [15] J. S. Dowker and K. Kirsten, *The Barnes zeta-function, sphere determinants and Glaisher–Kinkelin–Bendersky constants*, Analysis and Applications **3**(1) (2005), 45–68. doi:10.1142/S0219530505000479. Preprint: arXiv:hep-th/0301143v2.
- [16] W. Bridges, B. Brindle, K. Bringmann and J. Franke, *Asymptotic expansions for partitions generated by infinite products*, Mathematische Annalen **390** (2024), 2593–2632. doi:10.1007/s00208-024-02807-x.

TEACHER OF MATHEMATICS, THE CORNERSTONE ACADEMY, POOLE, UNITED KINGDOM