

FLOOR–CEILING TRANSFORMS OF PRIME COUNTING MEASURES

DHAIRYA SHAH

ABSTRACT. The floor–ceiling summation identities of Shah et al. are developed for prime and von Mangoldt measures. The main construction is a positive kernel obtained from the cubic floor–ceiling difference. We compute its ordinary Mellin transform, prove noncancellation at every nontrivial Riemann zero, separate the prime-square bias, and give an exponentially small truncation error after complete fourth-moment acceleration. A second prime-specific calculation identifies the logarithmic-density correction to rational scaling; the associated geometric series have the unit circle as a natural boundary. Exact weighted fibres, a scaled-hit classification, certified label stability and prime-root interpolation support these constructions. We also express Vasyunin’s classical biorthogonal system in reciprocal blocks and obtain the explicit lower bound $5/428$ for prime-only approximation scales, with the coefficient normalizations compared directly. Compact-window explicit formulas are applications of classical Mellin smoothing. Products, occupation transforms, local sieves and Fibonacci probes are collected as supporting applications in appendices. All needed rounded-count, interpolation, kernel and certificate arguments are supplied within the paper. The RH statements are error-bound criteria; their required global arithmetic estimates remain unproved.

1. INTRODUCTION

Floor and ceiling maps repeat output values over explicitly countable inverse intervals. The finite and infinite summation identities in [1, 2] exploit this repetition to transform power sums, geometric series, zeta functions and Fibonacci expressions. The present paper changes the input counting measure. Assigning weight $\mathbf{1}_{n \in \mathbb{P}}$ selects prime inputs; assigning weight $\Lambda(n)$ selects prime powers with their von Mangoldt weights. The resulting level multiplicities contain prime distribution and must be distinguished from the polynomial multiplicities obtained from all integer inputs.

The principal construction is the positive prime-measure trace of Sections 8–9. Starting with the cubic difference in the published framework, we obtain a kernel whose Mellin multiplier does not cancel any nontrivial Riemann zero. For this specified kernel we compute the prime-square bias and an explicit complete-moment evaluation bound. The same change of measure has a different effect on geometric traces: Theorem 6.2 calculates a logarithmic-density term that prevents the bounded scaling discrepancy familiar from integer counting. These kernel-specific formulas and the quantitative scale comparison are the focus of the paper.

The arithmetic ingredients are classical: the Euler product, Möbius inversion, the prime number theorem (PNT), Mellin inversion, and the Nyman–Beurling–Báez-Duarte criterion. Prime-zeta theory goes back to Fröberg [4]; analytic prime counting is treated, for example, by Lagarias and Odlyzko [5]. We use standard zeta-function estimates from [6, 3], and state the needed contour bounds below. The occupation construction is the usual colored-partition product [13]; its weights here are prime fibre counts. The statements concerning RH are equivalences or consequences of stated error bounds. They do not establish those global bounds or a new prime-gap theorem.

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1.1. Contributions and scope. The roles of the results are as follows.

- (1) *Specific prime-measure calculations.* Theorems 8.2, 9.1 and 9.2 combine a specified positive kernel, a zero-noncancellation calculation, the bias coefficient $K_d(1/2)/2$, and a complete fourth-moment correction with a positive exponential error bound. Theorem 6.2 gives the explicit second logarithmic term under scaling. Corollary 10.3 evaluates a two-vector certificate for prime-only dilation scales, giving $5/428$; no priority assertion is made for that numerical constant.
- (2) *Adaptations of the rounded-count framework.* Weighted inverse fibres, scaled integer-hit classification, all-root-order interval certificates and prime-root interpolation are proved directly in Sections 2–5. Their role is to track arithmetic weights and rounding errors exactly. The divided-difference principle is standard; the explicit prime-input interpolation domain and tail bound are recorded here.
- (3) *Classical mechanisms and supporting consequences.* The compact-window and positive-trace contour arguments implement the established explicit-formula method [6, 5]. The natural-boundary conclusion uses PNT and Pólya–Carlson after computing the prime-specific radial behavior. Section 10 identifies its dual functions exactly with Vasyunin’s biorthogonal system [9, 10] and translates the coefficient normalization used by Gallardo-Gutiérrez and Seco [11]. Squarefree-scale necessity is a classical coefficient-uniqueness consequence. Products, partitions, sieves and special-function probes in Appendices A–D illustrate the range of the prime fibre measure.

Thus the contribution is an explicit collection of prime-measure constructions and evaluation bounds, with its classical ingredients identified. It includes no improvement to a known prime-distribution error term. All supporting arguments are included.

Section 2 supplies the weighted identities, Sections 3–5 the exact and interpolation tools, and Section 6 the geometric analysis. Section 7 develops compact smoothing before the main positive-kernel results. Section 10 treats the prime-scale obstruction, and Section 11 records the computational scope and the remaining arithmetic estimates. The supplementary applications are retained with proofs in the appendices.

1.2. Notation. Write $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{P}$ denote the primes. For real $x \geq 0$ put

$$\pi(x) = \#\{p \leq x\}, \quad \theta(x) = \sum_{p \leq x} \log p, \quad \psi(x) = \sum_{n \leq x} \Lambda(n),$$

where $\Lambda(p^k) = \log p$ for $k \geq 1$ and $\Lambda(n) = 0$ otherwise. In particular $\Lambda(1) = 0$. A superscript minus, as in $\pi(t^-)$, denotes the left limit. All prime-counting endpoints are inclusive unless this notation is used. Set

$$D(s) = -\frac{\zeta'(s)}{\zeta(s)}, \quad P(s) = \sum_p p^{-s} \quad (\Re s > 1).$$

The letters ρ denote distinct nontrivial zeros of ζ , and m_ρ their multiplicities. We use

$$\Theta = \sup\{\Re \rho : \zeta(\rho) = 0, \ 0 < \Re \rho < 1\}.$$

The functional equation and the existence of critical-line zeros imply $1/2 \leq \Theta \leq 1$. Complex powers of positive real numbers use the real logarithm. All holomorphic-series assertions below include normal convergence on compact subsets of the stated domain.

2. LEVEL COUNTS AND WEIGHTED TRANSFORMATIONS

For $a, \beta > 0$, let

$$H_F(n) = \lfloor (\beta n)^a \rfloor, \quad H_C(n) = \lceil (\beta n)^a \rceil, \quad r = 1/a, \quad \alpha = 1/\beta.$$

These are the rounded power maps underlying [1, Theorems 1–2] and [2, Theorems 1–2]. The following derivation includes arbitrary input weights and both partial endpoint blocks.

Proposition 2.1 (Exact inverse fibres). *For integers $1 \leq m \leq N$, the floor fibre of level $j \geq 0$ has integer endpoints*

$$L_j^F = \max\{m, \lceil \alpha j^r \rceil\}, \quad U_j^F = \min\{N, \lceil \alpha(j+1)^r \rceil - 1\},$$

and the ceiling fibre of level $j \geq 1$ has endpoints

$$L_j^C = \max\{m, \lfloor \alpha(j-1)^r \rfloor + 1\}, \quad U_j^C = \min\{N, \lfloor \alpha j^r \rfloor\}.$$

A fibre is empty when its upper endpoint is smaller than its lower endpoint. If $\Phi(t) = \sum_{m \leq n \leq t} \omega_n$, its weight on a nonempty fibre is $\Phi(U_j) - \Phi(L_j - 1)$.

Proof. The floor condition is $j \leq (\beta n)^a < j+1$, equivalent to $\alpha j^r \leq n < \alpha(j+1)^r$. The ceiling condition is $j-1 < (\beta n)^a \leq j$, equivalent to $\alpha(j-1)^r < n \leq \alpha j^r$. Count the integers in these intervals after intersecting with $[m, N]$, and subtract their weighted prefixes. $\square$

Proposition 2.2 (Weighted sums and products). *Let S be finite, $H : S \rightarrow J$ any map and $W_j = \sum_{n \in S: H(n)=j} \omega_n$. Then*

$$(2.1) \quad \sum_{n \in S} \omega_n u_{H(n)} = \sum_j W_j u_j.$$

For integer weights and nonzero factors when negative powers occur,

$$\prod_{n \in S} v_{H(n)}^{\omega_n} = \prod_j v_j^{W_j}.$$

For complex weights the product identity holds when both sides are defined by exponentiating the sums of the same specified logarithms. These identities compose: the weight at a further output $\ell = g(j)$ is $\sum_{g(j)=\ell} W_j$.

Proof. The disjoint sets $H^{-1}(j)$ partition S . Collect the terms or factors on each set. Applying this partition a second time proves the composition statement. With specified logarithms the product assertion is the additive assertion followed by exponentiation. $\square$

Thus prime multiplicities and Mangoldt multiplicities are obtained by taking $\omega_n = \mathbf{1}_{n \in \mathbb{P}}$ and $\omega_n = \Lambda(n)$, respectively. In particular the full prime fibres are

$$(2.2) \quad \begin{aligned} b_j^F &= \pi((\alpha(j+1)^r)^-) - \pi((\alpha j^r)^-), & j \geq 0, \\ b_j^C &= \pi(\alpha j^r) - \pi(\alpha(j-1)^r), & j \geq 1. \end{aligned}$$

The floor-zero level is finite and is retained whenever the kernel is defined there. A kernel singular at zero requires its explicit removal.

For infinite sums, nonnegative terms permit the identities by monotone convergence. Absolute convergence permits them by Tonelli's theorem applied to absolute values. The following ordered version explains the additional condition for signed weights.

Lemma 2.3 (Incomplete blocks). *Suppose a nondecreasing unbounded map has finite consecutive fibres B_j , in increasing occupied-level order. Put*

$$W_j = \sum_{n \in B_j} \omega_n, \quad M_j = \max_{\text{initial } I \subseteq B_j} \left| \sum_{n \in I} \omega_n \right|.$$

The ordered input sum $\sum_n \omega_n u_{H(n)}$ converges if and only if $\sum_j W_j u_j$ converges and $|u_j| M_j \rightarrow 0$. For nonnegative weights the second condition follows from convergence of the grouped series. If products are defined using specified logarithms, the same assertion applied to $u_j = \log v_j$ is a sufficient criterion for a nonzero product limit.

Proof. At complete-block endpoints the two partial sums agree. A partial sum inside block j differs from its preceding endpoint by u_j times an initial weighted sum in that block. Its maximal modulus is $|u_j| M_j$. This proves sufficiency. Convergence of all input partial sums makes every sufficiently late within-block tail uniformly small, proving necessity. For

nonnegative weights $M_j = W_j$, and convergence of the grouped series implies $W_j u_j \rightarrow 0$. Exponentiation proves the logarithmic product statement. $\square$

Convergence of endpoint *products* alone is insufficient. For example, a block of size $d_j = 2j - 1$ with factor $\exp(2\pi i/d_j)$ has full product one, while a half-block product tends to -1 . In general a nonzero product limit is equivalent to a nonzero endpoint-product limit together with convergence to one of every within-block multiplicative remainder, uniformly over its positions. This follows by dividing each partial product by its preceding endpoint.

2.1. Unit integer roots and the original zeta pair. For $q \in \mathbb{N}$ write

$$h_{F,q}(n) = \lfloor n^{1/q} \rfloor, \quad h_{C,q}(n) = \lceil n^{1/q} \rceil.$$

Here and below $h_{X,q}$, for $X \in \{F, C\}$, denotes the corresponding rounded map. The letters f_q and c_q denote the multiplicity polynomials below. Proposition 2.1 with unit scale gives the unweighted multiplicities

$$(2.3) \quad f_q(j) = (j+1)^q - j^q, \quad c_q(j) = j^q - (j-1)^q \quad (j \geq 1).$$

Define $F_q(s) = \sum_{j \geq 1} f_q(j)j^{-s}$ and $C_q(s) = \sum_{j \geq 1} c_q(j)j^{-s}$ for $\Re s > q$.

Proposition 2.4 (Shifted-zeta and heat identities). *The functions have meromorphic continuations*

$$(2.4) \quad \begin{aligned} F_q(s) &= \sum_{k=0}^{q-1} \binom{q}{k} \zeta(s-k), \\ C_q(s) &= \sum_{k=0}^{q-1} (-1)^{q-k+1} \binom{q}{k} \zeta(s-k). \end{aligned}$$

In particular,

$$(2.5) \quad F_2 - C_2 = 2\zeta(s), \quad F_3 - C_3 = 6\zeta(s-1).$$

For $t > 0$, the corresponding positive-level heat traces satisfy

$$(2.6) \quad \sum_{j \geq 1} (f_3(j) - c_3(j))e^{-jt} = \frac{6e^{-t}}{(1-e^{-t})^2}.$$

Proof. Expand (2.3) by the binomial theorem and sum absolutely in $\Re s > q$. Each monomial gives a shifted zeta function, proving (2.4) and its continuation. For $q = 2, 3$ subtraction gives (2.5). Finally $f_3(j) - c_3(j) = 6j$ and differentiating $\sum_{j \geq 1} z^j = z/(1-z)$ gives (2.6). This also proves directly the specializations of [2, Corollaries 3–4] needed here. $\square$

Remark 2.5 (Kernels and prime information). Every finite power, geometric, telescoping or Fibonacci expression in [1], and every convergent zeta, Hurwitz, Lerch, polylogarithmic or reciprocal-Fibonacci expression in [2], can be weighted by (2.1). For example, the simultaneous kernel is $\sum_j b_j z^j (j+y)^{-s}$, and a finite product is $\prod_j v(j)^{b_j}$. Pascal's identity gives

$$\sum_{p \leq N} ((h_{F,q}(p) + 1)^m - h_{F,q}(p)^m) = \sum_{\ell < m} \binom{m}{\ell} \sum_{p \leq N} h_{F,q}(p)^\ell.$$

These are exact weighted consequences, not ordinary Faulhaber polynomials for prime counts. Selecting prime *output labels* in an unweighted identity is a different operation from selecting prime *inputs*.

3. PRIME EXTRACTION AND INTEGER HITS

Proposition 3.1 (Prime-power removal). *For every function u on integers at least two and finite $Y \geq 2$,*

$$(3.1) \quad \sum_{p \leq Y} u(p) = \sum_{k \leq \log_2 Y} \frac{\mu(k)}{k} \sum_{2 \leq n \leq Y^{1/k}} \frac{\Lambda(n)}{\log n} u(n^k),$$

$$(3.2) \quad \sum_{p \leq Y} \log p u(p) = \sum_{k \leq \log_2 Y} \mu(k) \sum_{2 \leq n \leq Y^{1/k}} \Lambda(n) u(n^k).$$

Moreover,

$$(3.3) \quad \psi(Y) = \sum_{d \leq Y} \mu(d) \log \Gamma(\lfloor Y/d \rfloor + 1).$$

Proof. Only $n = p^r$ contributes. At a fixed exponent $m = rk$, the coefficient in (3.1) is $m^{-1} \sum_{k|m} \mu(k)$, equal to one for $m = 1$ and zero otherwise. In (3.2) the same divisor sum multiplies $\log p$. Finally $\Lambda = \mu * \log$ follows by applying Möbius inversion to $\log n = \sum_{d|n} \Lambda(d)$. Sum this convolution through Y and use $\sum_{m \leq Y/d} \log m = \log(\lfloor Y/d \rfloor!)$. All rearrangements are finite. $\square$

Thus $J(Y) = \sum_{p^r \leq Y} 1/r$ satisfies $\pi(Y) = \sum_{k \leq \log_2 Y} \mu(k) J(Y^{1/k})/k$, and $\theta(Y) = \sum_{k \leq \log_2 Y} \mu(k) \psi(Y^{1/k})$. Differences of (3.3) at U and $L - 1$ give the exact Mangoldt weight of every clipped fibre. Exponentiation yields

$$\prod_{d \leq N} (\lfloor N/d \rfloor!)^{\mu(d)} = e^{\psi(N)} = \prod_{p \leq N} p^{\lfloor \log_p N \rfloor} = \text{lcm}(1, \dots, N).$$

This is a finite product identity, not a complexity improvement for prime counting.

The Euler product, differentiated in $\Re s > 1$, gives $D(s) = \sum_n \Lambda(n) n^{-s}$. Proposition 2.4 therefore recovers it from the published pair:

$$(3.4) \quad D(s) = -\frac{F'_2(s) - C'_2(s)}{F_2(s) - C_2(s)}.$$

Expanding Euler logarithms and applying the same divisor sum gives

$$(3.5) \quad P(s) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log \zeta(ks) \quad (\Re s > 1).$$

These logarithms are the holomorphic Euler-product logarithms. This classical identity [4] is not an unrestricted principal-log formula across zeta zeros. Perron inversion of $D(s)/s$ gives the midpoint value of ψ at a jump; a right limit recovers the inclusive convention. Applying the finite inversion above then gives θ and π ; see [5, 6].

3.1. Unit-scale prime blocks. For $q \geq 2$ and prime p , the root $p^{1/q}$ is not integral: $p = j^q$ would contradict primality. Hence $h_{C,q}(p) = h_{F,q}(p) + 1$, and

$$(3.6) \quad b_{q,j} = \pi((j+1)^q) - \pi(j^q), \quad G_{C,q}(z) = z G_{F,q}(z),$$

where $G_{F,q}(z) = \sum_p z^{h_{F,q}(p)} = \sum_{j \geq 1} b_{q,j} z^j$. Both endpoints of each prime block are nonprime. In particular,

$$\sum_{j=1}^J b_{q,j} = \pi((J+1)^q), \quad \pi(J^q) = [z^{J-1}] \frac{G_{F,q}(z)}{1-z}.$$

For $q = 2$, positivity of every $b_{2,j}$ is precisely Legendre's conjecture. The PNT gives cumulative and long-average asymptotics, such as

$$\sum_{j=J+1}^{2J} b_{q,j} \sim \frac{2^q - 1}{q} \frac{J^q}{\log J},$$

but not a pointwise asymptotic in every square block. Eventual positivity for each fixed $q \geq 3$ already follows from the public short-interval theorem of Baker, Harman and Pintz [7]: apply its exponent 0.525 at $(j+1)^q$ and use $q-1 > 0.525q$. This observation is separate from the square-block conjecture.

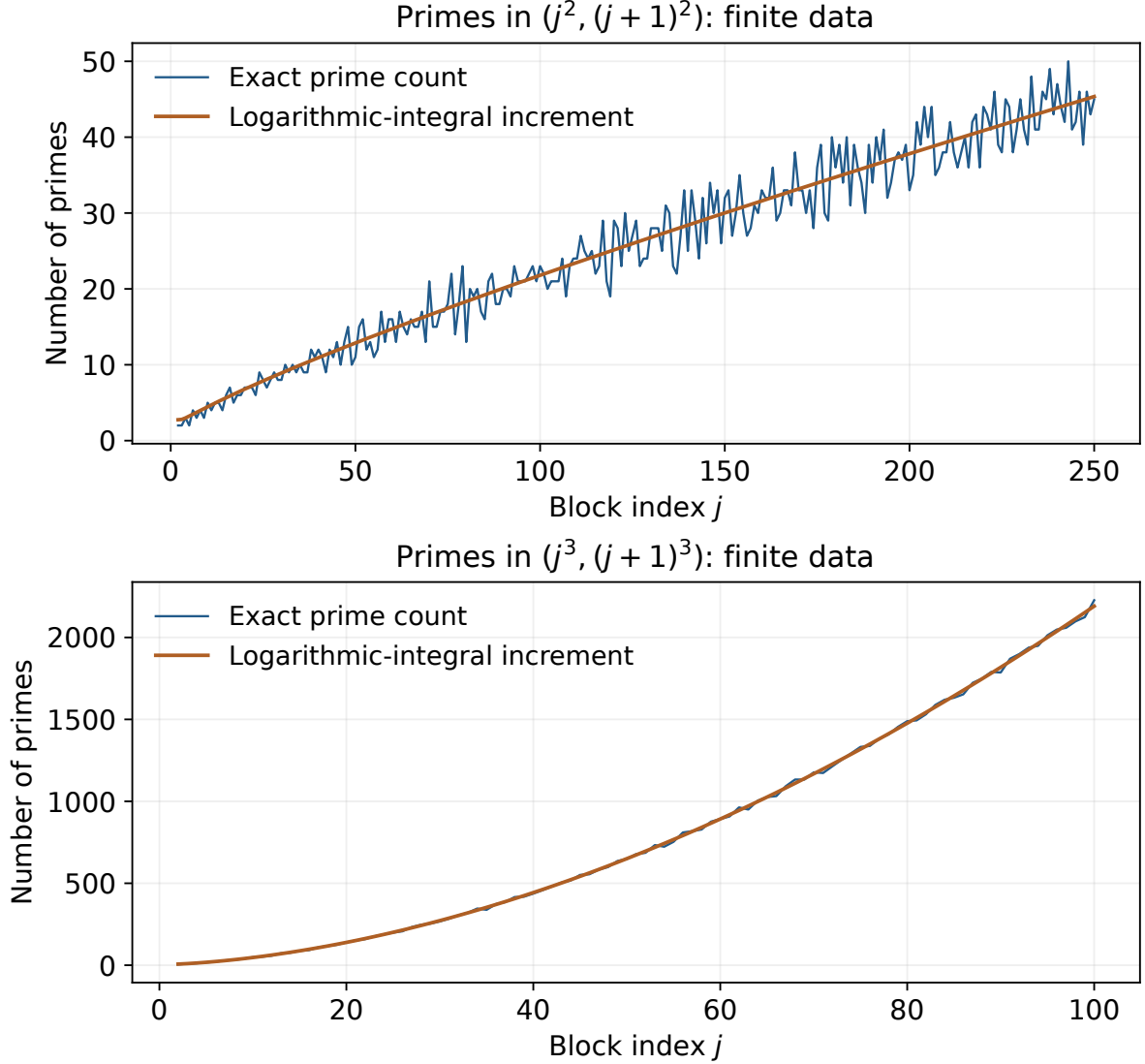


FIGURE 1. Exact finite prime counts between consecutive squares and cubes, with logarithmic-integral increments for reference. The reference curves are not certified pointwise approximations.

3.2. The general integer-hit identity. Put $\tau(t) = 1 + \lfloor t \rfloor - \lceil t \rceil$, the indicator of the integers. Define the scaled rounded maps $h_{F,q,\beta}(n) = \lfloor (\beta n)^{1/q} \rfloor$ and $h_{C,q,\beta}(n) = \lceil (\beta n)^{1/q} \rceil$.

Lemma 3.2 (Weighted hit cancellation). *Whenever the following sums converge absolutely, for $0 < z < 1$,*

$$(3.7) \quad \sum_n \omega_n z^{h_{C,q,\beta}(n)} - z \sum_n \omega_n z^{h_{F,q,\beta}(n)} = (1-z) \sum_{\substack{n \geq 1 \\ (\beta n)^{1/q} \in \mathbb{Z}}} \omega_n z^{(\beta n)^{1/q}}.$$

For unit-scale Mangoldt weights put $\mathcal{L}_{X,q}(z) = \sum_n \Lambda(n) z^{h_{X,q}(n)}$, $X = F, C$. In particular, for $q \geq 2$,

$$(3.8) \quad \mathcal{L}_{C,q}(z) - z\mathcal{L}_{F,q}(z) = (1-z)\mathcal{L}(z), \quad \mathcal{L}(z) = \sum_n \Lambda(n) z^n.$$

Proof. A nonintegral root has ceiling one more than floor, so its contribution vanishes. An integral root j contributes $(1-z)\omega_n z^j$. At unit scale the hits are $n = j^q$ and $\Lambda(j^q) = \Lambda(j)$, proving (3.8). Absolute convergence follows for these weights from the exponential level decay and polynomial bounds on the fibre masses. $\square$

Theorem 3.3 (All scaled prime and prime-power hits). *Fix $q \geq 2$ and $\beta > 0$. There is at most one prime p with $\beta p = j^q$ for an integer j . Such a hit forces β to be rational. Writing $\beta = u/v$ in lowest terms, the possibilities are exactly*

$$v = p \text{ prime and } u = A^q, \quad \text{or} \quad v = 1 \text{ and } u = A^q p^{q-1}, \quad A \in \mathbb{N}.$$

For Mangoldt weights let

$$\mathcal{H}_{q,\beta}(z) = \sum_{\substack{n \geq 2 \\ (\beta n)^{1/q} \in \mathbb{Z}}} \Lambda(n) z^{(\beta n)^{1/q}}.$$

If $\beta = A^q$ for an integer $A \geq 1$, then $\mathcal{H}_{q,\beta}(z) = \mathcal{L}(z^A)$. Otherwise at most one prime base ℓ contributes. If it does, there are $C \in \mathbb{N}$, $(C, \ell) = 1$, and $e \in \mathbb{Z}$ with

$$(3.9) \quad \beta = C^q \ell^e, \quad m_0 = \max\{0, \lceil (e+1)/q \rceil\}, \quad \mathcal{H}_{q,\beta}(z) = \log \ell \sum_{m \geq m_0} z^{C \ell^m}.$$

If no such base exists, or if β is irrational, the hit trace is zero.

Proof. Two prime hits would give $p/r = (j/k)^q$ for distinct primes. Its valuation at p is one, which is not a multiple of q . Reducing j^q/p gives either denominator p and a perfect-power numerator, or the integer $A^q p^{q-1}$. These forms also give the converse.

For a rational scale write $e_r = v_r(\beta)$. A power $n = \ell^k$, $k \geq 1$, is a hit exactly when

$$e_r \in q\mathbb{Z}_{\geq 0} \ (r \neq \ell), \quad e_\ell + k \in q\mathbb{Z}_{\geq 0}.$$

If two bases are possible these tests force every valuation of β to be nonnegative and divisible by q ; hence $\beta = A^q$. Then k is divisible by q and the hit level is $A \ell^{k/q}$, which gives $\mathcal{L}(z^A)$. In the remaining case all valuations except possibly that at the one base form C^q . Write $e + k = qm \geq 0$. The inequality $k \geq 1$ is exactly $m \geq m_0$, proving (3.9). $\square$

For example $q = 2$, $\beta = 2$ has only the prime hit $p = 2$, but prime-power hits $n = 2^{2m-1}$ at levels 2^m , $m \geq 1$. At $\beta = 4$, every prime base contributes to $\mathcal{L}(z^2)$. Formula (3.7) gives the complete boundary correction in either case.

There is a related limitation of prime-index rationality tests. If $x = u/v$ is reduced, then $\tau(px) = 1$ exactly when $v \mid p$. Consequently

$$\sum_p \tau(px) z^p = \begin{cases} \sum_p z^p, & x \in \mathbb{Z}, \\ z^v, & v \text{ prime}, \\ 0, & v > 1 \text{ composite or } x \notin \mathbb{Q}. \end{cases}$$

The composite denominator 6 is therefore indistinguishable from an irrational input by this particular prime-index series. This does not obstruct the finite certificate developed next.

4. CERTIFIED STABILITY OF ROUNDED SCALES

For rational $L < U$, define

$$H_{[L,U]}(v) = \lfloor vU \rfloor - \lfloor vL \rfloor + 1, \quad q_{\min}([L,U]) = \min\{v \geq 1 : H_{[L,U]}(v) > 0\}.$$

The first quantity counts candidate numerators in $[vL, vU]$. The second exists because intervals of length at least one contain an integer. A first candidate denominator is already reduced, so $q_{\min}$ is exactly the minimum reduced rational denominator in $[L, U]$.

Lemma 4.1 (Adjacent-fraction certificate). *Suppose $b, d > 0$, $bc - ad = 1$ and*

$$(4.1) \quad \frac{a}{b} < L \leq \frac{a+c}{b+d} \leq U < \frac{c}{d}.$$

Then $q_{\min}([L, U]) = b + d$.

Proof. For any r/s strictly between the parents, $cs - dr$ and $br - as$ are positive integers, and

$$s = b(cs - dr) + d(br - as) \geq b + d.$$

The mediant attains the bound and is reduced, since a common divisor of $a + c$ and $b + d$ divides $b(a + c) - a(b + d) = 1$. This proves minimality without an exhaustive denominator search. $\square$

Theorem 4.2 (One certificate for every integer root order). *If $0 < L \leq \beta \leq U$ and $q_{\min}([L, U]) > N$, then for every $n \leq N$ and every integer $q \geq 1$, both $h_{F,q,\beta}(n)$ and $h_{C,q,\beta}(n)$ are constant as β varies over the entire closed interval. No integer hit occurs there. Every finite weighted transform through input N consequently has identical rounded labels throughout the interval.*

Proof. A label can change only at a scale $\beta = j^q/n$ with $j \geq 1$. Its reduced denominator is at most $n \leq N$, contradicting the certificate. Thus no threshold occurs, including at either endpoint. Continuity of $(\beta n)^{1/q}$ fixes its two rounded labels on the connected interval. Equality of all labels proves the weighted assertion by Proposition 2.2. $\square$

Example 4.3. Let $B = 10^{16}$ and $l = 14142135623730950$. Exact integer squaring verifies $l^2 < 2B^2 < (l+1)^2$, so $\sqrt{2}$ belongs to $[L, U] = [l/B, (l+1)/B]$. Exact comparisons give

$$\frac{54608393}{38613965} < L \leq \frac{131836323}{93222358} \leq U < \frac{77227930}{54608393}.$$

The determinant of the parents is one. Lemma 4.1 gives $q_{\min} = 93\,222\,358$. Thus every input through $93\,222\,357$, and every integer root order, is protected. The rational mediant is an interval candidate, not an identification of it with $\sqrt{2}$.

For completeness, the certificate can be found by a finite compressed Stern–Brocot search. If $\lceil L \rceil \leq U$, the minimum is one. Otherwise subtract $h = \lfloor L \rfloor$, write $L - h = \lambda/\rho$, $U - h = \mu/\sigma$ in $(0, 1)$, and start with parents $(a, b, c, d) = (0, 1, 1, 0)$. A mediant below the lower endpoint permits the left jump

$$k = \left\lfloor \frac{\lambda b - a\rho - 1}{c\rho - \lambda d} \right\rfloor, \quad (a, b) \leftarrow (a + kc, b + kd).$$

A mediant above the upper endpoint permits the right jump

$$k = \left\lfloor \frac{c\sigma - \mu d - 1}{\mu b - a\sigma} \right\rfloor, \quad (c, d) \leftarrow (c + ka, d + kb).$$

The denominators are positive by strict bracketing, and the maximal permissible integer jump is exactly the displayed floor; the subtraction of one preserves strict inequalities at rational endpoints. The determinant remains one. To prove termination, fix a rational r/s in the target interval. Its positive coordinates $cs - dr$, $br - as$ remain positive while it is bracketed. Each left jump subtracts a positive multiple of the first coordinate from the second; each right

jump does the converse. Their sum decreases, so only finitely many jumps are possible. The returning median supplies (4.1), after shifting by h .

A finite enclosure alone does not determine rationality: every interval of positive width contains rational and irrational numbers. The certificate instead excludes precisely the rational thresholds that could change the specified finite calculation. When it is unavailable, monotonicity still gives, for $0 < z < 1$,

$$\sum_{p \leq N} z^{h_{X,q,U}(p)} \leq \sum_{p \leq N} z^{h_{X,q,\beta}(p)} \leq \sum_{p \leq N} z^{h_{X,q,L}(p)}, \quad X = F, C.$$

5. PRIME-ROOT DIRICHLET TRANSFORMS AND INTERPOLATION

For $q \geq 2$ define

$$Q_{F,q}(s) = \sum_p h_{F,q}(p)^{-s}, \quad Q_{C,q}(s) = \sum_p h_{C,q}(p)^{-s} \quad (\Re s > q).$$

These prime-input functions are different from the unweighted F_q, C_q of Proposition 2.4.

Theorem 5.1 (Holomorphic rounding error). *For $X = F, C$ there is a holomorphic $R_{X,q}$ on $\Re s > q - 1$ such that*

$$Q_{X,q}(s) = P(s/q) + R_{X,q}(s), \quad R_{X,q}(s) = \sum_p (h_{X,q}(p)^{-s} - p^{-s/q}).$$

The first identity initially holds in $\Re s > q$; the second is normally convergent in its stated larger domain.

Proof. Put $u = p^{1/q}$. For $u \geq 2$, the rounded argument and every point between it and u lie between $u/2$ and $2u$. The identity

$$v^{-s} - u^{-s} = -s \int_u^v t^{-s-1} dt$$

bounds the difference by $O_K(p^{-(\sigma+1)/q})$, uniformly on a compact set K of $s = \sigma + it$. The full integer majorant converges when $\sigma > q - 1$. The finitely many smaller primes contribute entire functions. $\square$

In the square-root case, the initial exponential has a useful single-valued continuation.

Corollary 5.2. *On $\Re s > 1$ the continuation of $\exp Q_{F,2}(s)$ is*

$$\mathcal{E}_{F,2}(s) = \zeta(s/2)A_F(s), \quad A_F(s) = \exp \left(R_{F,2}(s) + \sum_{k \geq 2} \frac{\mu(k)}{k} \log \zeta(ks/2) \right),$$

where A_F is holomorphic and zero-free. Its zeros in that half-plane are exactly 2ρ with $\Re \rho > 1/2$, with multiplicities, and its only pole is at two. Hence

$$\text{RH} \iff (s-2)\mathcal{E}_{F,2}(s) \text{ is holomorphic and zero-free on } \Re s > 1.$$

The ceiling transform has the same properties.

Proof. For $k \geq 2$ and $\Re s > 1$, each $ks/2$ lies in the Euler-product half-plane. On compact subsets the Euler logarithms decay exponentially with k , so their series converges normally. Exponentiating (3.5) with Theorem 5.1 proves the formula initially on $\Re s > 2$ and then its meromorphic continuation. The nonzero exponential factor cannot cancel a zero or pole of $\zeta(s/2)$. Reflection of the zeta zero set proves the equivalence. This continues the initial exponential; it does not assert a holomorphic logarithm at its continued zeros. $\square$

Lemma 5.3 (Interpolation remainder). *Let $j \geq 1$, $0 \leq \delta \leq 1$, and*

$$\ell_\nu(\delta) = \prod_{\substack{0 \leq r < L \\ r \neq \nu}} \frac{\delta - r}{\nu - r}, \quad 0 \leq \nu < L.$$

For $L \geq 1$ and $\sigma + L > 0$,

$$(5.1) \quad \left| \sum_{\nu < L} \ell_\nu(\delta)(j + \nu)^{-s} - (j + \delta)^{-s} \right| \leq B_L |(s)_L| j^{-\sigma-L},$$

where $B_1 = 1$, $B_L = 1/(4L)$ for $L \geq 2$, and $(s)_L = s(s+1) \cdots (s+L-1)$.

Proof. The real-node divided-difference remainder is the product of the distances from the nodes times an integral of the L th derivative over a simplex of mass $1/L!$. This formula follows by repeated use of the fundamental theorem of calculus, and holds for complex-valued functions by applying it to their real and imaginary parts. The derivative of v^{-s} is $(-1)^L (s)_L v^{-s-L}$, whose modulus is at most $|(s)_L| j^{-\sigma-L}$ on the relevant interval. For $L = 1$ the distance is at most one. For $L \geq 2$ the node product is bounded by $\delta(1-\delta) \prod_{r=2}^{L-1} r \leq (L-1)!/4$. Division by $L!$ proves the bound. $\square$

Theorem 5.4 (Higher interpolation). *With $j_p = h_{F,q}(p)$ and $\delta_p = p^{1/q} - j_p$, set*

$$Q_{q,L}^{\text{int}}(s) = \sum_p \sum_{\nu < L} \ell_\nu(\delta_p)(j_p + \nu)^{-s}.$$

There is a holomorphic $R_{q,L}$ on $\Re s > q - L$ such that $Q_{q,L}^{\text{int}} = P(s/q) + R_{q,L}$ initially on $\Re s > q$. If its difference series is cut after complete block J , then, for $\eta = \sigma + L - q > 0$,

$$(5.2) \quad |R_{q,L}(s) - R_{q,L}^{[J]}(s)| \leq \frac{(2^q - 1)B_L |(s)_L|}{\eta} J^{-\eta}.$$

Proof. There are at most $f_q(j) \leq (2^q - 1)j^{q-1}$ integer inputs in block j . Multiply this bound by (5.1) and sum. The exponent is $-1 - \eta$, proving normal convergence. Comparing the remaining sum with its integral from J proves (5.2). The interpolation weights reproduce constants, so the leading prime-zeta term has coefficient one. $\square$

6. GEOMETRIC TRANSFORMS, NATURAL BOUNDARIES AND SCALING

For the general positive powers and scales of Section 2 let $G_X(z) = \sum_p z^{H_X(p)}$, retaining a finite floor-zero term if present.

Theorem 6.1 (Prime geometric boundary). *Both functions have radius of convergence one and the unit circle as a natural boundary. As $t \downarrow 0$,*

$$(6.1) \quad G_X(e^{-t}) \sim \frac{\alpha \Gamma(r)}{t^r \log(1/t)}, \quad r = 1/a, \quad \alpha = 1/\beta.$$

Consequently no nonzero polynomial multiplier makes either function rational, and its coefficients have no eventual constant-coefficient linear recurrence or quasipolynomial description.

Proof. The cumulative floor count through level J is $\pi((\alpha(J+1)^r)^-)$, and by the PNT it is asymptotic to $(\alpha/r)J^r/\log J$. Its coefficients are nonnegative integers with polynomial growth and infinitely many nonzero terms; these facts give radius exactly one.

Here is the Abelian step explicitly. If $A(y)$ is the cumulative level count, then $G_F(e^{-t}) = t \int_0^\infty e^{-ty} A(y) dy$. Put $v = ty$ and $L_t = \log(1/t)$. For fixed $v > 0$, $t^r L_t A(v/t) \rightarrow (\alpha/r)v^r$. For $v \geq \sqrt{t}$, the PNT upper bound gives $t^r L_t A(v/t) \leq Cv^r$, since $\log(v/t) \geq L_t/2$. For $0 < v < \sqrt{t}$, the global bound $A(y) \leq C(1 + y^r)$ gives an integral tending to zero after normalization. Dominated convergence on the remaining range proves the limit $(\alpha/r)\Gamma(r+1) = \alpha\Gamma(r)$. The inequalities $e^{-t}G_F(e^{-t}) \leq G_C(e^{-t}) \leq G_F(e^{-t})$ give the ceiling limit.

A rational function divergent at $z = 1$ has an integer-order pole with a nonzero pure-power leading term. The logarithmic factor in (6.1) precludes that form for every $r > 0$. The classical Pólya–Carlson theorem for integer-coefficient series of radius one [12] now gives the natural boundary. A polynomial multiplier giving a rational function would make G_X rational. An eventual linear recurrence gives a rational generating function, and an eventual quasipolynomial is a special case. $\square$

Theorem 6.2 (The logarithmic-density scale correction). *For fixed integer $q \geq 1$ and $\beta > 0$, let $G_{X,q,\beta}(e^{-t}) = \sum_p e^{-th_{X,q,\beta}(p)}$. With $\Psi = \Gamma'/\Gamma$ and $L_t = \log(1/t)$,*

$$(6.2) \quad G_{X,q,\beta}(e^{-t}) = \frac{\Gamma(q)}{\beta t^q} \left(\frac{1}{L_t} + \frac{(\log \beta)/q - \Psi(q)}{L_t^2} + O_{q,\beta}(L_t^{-3}) \right).$$

Thus for $\beta \neq 1$,

$$(6.3) \quad \beta^{-1} G_{X,q,1}(e^{-t}) - G_{X,q,\beta}(e^{-t}) \sim -\frac{\Gamma(q) \log \beta}{q \beta t^q L_t^2}.$$

Proof. Use the public PNT estimate $\pi(x) - \text{li}(x) = O(xe^{-c\sqrt{\log x}})$ [3, §27.12]. Integration by parts against $e^{-t(\beta x)^{1/q}}$, split at $x = t^{-q/2}$, makes this replacement error smaller than $t^{-q} L_t^{-3}$. The initial interval contributes at most $O(t^{-q/2})$; on the remaining interval the exponential PNT factor dominates every fixed inverse power of L_t .

For the integral with $dx/\log x$, put $x = u^q/\beta$ and $v = tu$. It becomes

$$\frac{1}{\beta t^q} \int_{t(2\beta)^{1/q}}^{\infty} \frac{v^{q-1} e^{-v}}{L_t + \log v - (\log \beta)/q} dv.$$

On $\sqrt{t} \leq v \leq t^{-1/2}$ expand the denominator through its linear correction. The remainder is bounded by $CL_t^{-3}(1 + |\log v|^2)$ against the integrable weight $v^{q-1} e^{-v}$. The smaller range has contribution $O(t^{q/2})$ before the factor t^{-q} , and the larger range is exponentially small. The first two logarithmic moments are $\Gamma(q)$ and $\Gamma'(q)$, which prove the smooth-root expansion.

Rounding changes a summand by at most $(e^t - 1)e^{-t(\beta p)^{1/q}}$. The total is $O(t^{1-q}/L_t)$ by (6.1) and the corresponding smooth-root comparison. This is negligible against $t^{-q} L_t^{-3}$. Subtract the expansions at the two fixed scales to obtain (6.3). $\square$

For comparison, the unweighted ceiling count obeys an error of less than one in its cumulative comparison with interval length. Summation by parts transfers that bounded discrepancy to decreasing kernels. It does not transfer to prime measure: (6.3) is unbounded and is eventually negative for $\beta > 1$.

Exact set complementation can still survive prime restriction. If $\alpha > 1$ is irrational and $\beta = \alpha/(\alpha - 1)$, then $j/\alpha + j/\beta = j$ and neither summand is an integer. Their ceilings sum to $j + 1$; subtracting at consecutive j proves that the two Beatty sequences $[\alpha n]$ and $[\beta n]$ partition $\mathbb{N}$. Multiplication of their indicators by $\mathbf{1}_{m \in \mathbb{P}}$ gives $\pi_\alpha(Y) + \pi_\beta(Y) = \pi(Y)$. A bounded allocation discrepancy need not survive: for $u_m = \lfloor (m+1)/2 \rfloor - \lfloor m/2 \rfloor$ and $Y \geq 2$,

$$\sum_{p \leq Y} (u_p - 1/2) = \pi(Y)/2 - 1.$$

This follows because every odd prime has $u_p = 1$, while $u_2 = 0$. Equidistribution assertions for irrational allocations require separate information about fractional parts along primes.

7. COMPACT ROOT WINDOWS AND THE ZETA ZEROS

The following construction is classical Mellin smoothing of the Mangoldt measure, expressed in the integer-root variable. Its purpose here is to identify the smoothing multiplier, the literal-rounding reference terms and the interpolation loss precisely. The RH equivalence follows from the standard explicit-formula and Mellin-pole arguments [6, 5].

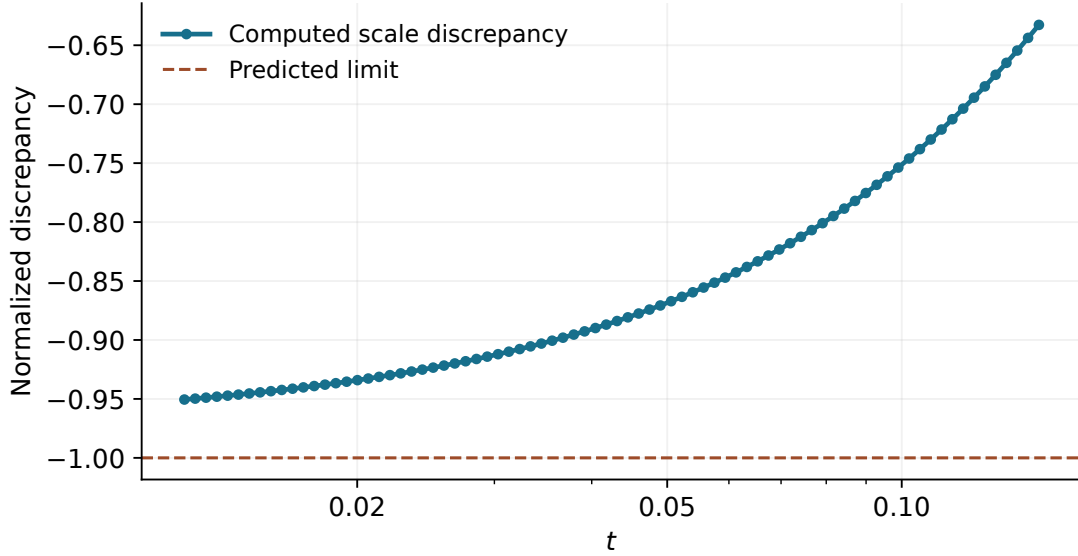


FIGURE 2. Ceiling square-root traces at scales one and two. The discrepancy in (6.3) is normalized to have predicted limit -1 . The finite range illustrates the density correction; it is not an effective remainder estimate.

We record explicitly the classical contour input used in both smoothing constructions. Besides the Euler product and functional equation, we use the completed-zeta Hadamard product and the Riemann–von Mangoldt zero-count estimate; these standard public results are given in [6, Chapters 2 and 9].

Lemma 7.1 (Logarithmic-derivative contour bounds). *On a fixed bounded horizontal strip of real parts, for $|t| \geq 3$ and away from zeros,*

$$(7.1) \quad \frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\Im \rho - t| \leq 1} \frac{m_\rho}{\sigma + it - \rho} + O(\log(2 + |t|)).$$

There are $O(\log(2 + |t|))$ zeros, counted with multiplicity, in the indicated interval. Consequently $D(s) = O_\delta(\log(2 + |t|))$ on each fixed bounded strip with $\sigma \geq \Theta + \delta$. Horizontal heights tending to infinity can be chosen on which $D(s) = O(\log^2(2 + |t|))$ uniformly across a fixed bounded real strip. On any fixed vertical line of negative real part avoiding the real poles, $D(s) = O(\log(2 + |t|))$.

Proof. Subtract the logarithmic derivative of the completed-zeta Hadamard product at s and at $b + it$, where fixed $b > 1$ lies to the right of the real strip under consideration. For zeros with $|\Im \rho - t| > 1$, the difference of reciprocals is bounded by a constant times $(1 + |\Im \rho - t|)^{-2}$. Summation using the local zero count is $O(\log(2 + |t|))$. The finitely many nearer reference reciprocals are bounded because $b - \Re \rho \geq b - 1 > 0$. The gamma and elementary factors in the completed function contribute $O(\log(2 + |t|))$ by Stirling’s formula, while the Euler series bounds $D(b + it)$. This gives (7.1). The local zero count follows by differencing the Riemann–von Mangoldt formula over a bounded interval.

If $\sigma \geq \Theta + \delta$, each displayed denominator has modulus at least δ , proving the first bound. In each sufficiently large unit-height interval discard neighborhoods of radius $c/\log(2 + |t|)$ around the $O(\log(2 + |t|))$ zero ordinates in its fixed enlargement. Choosing c small leaves an admissible height; (7.1) gives the stated quadratic-log bound there. In the left half-plane the logarithmic derivative of the functional equation reduces the assertion to the convergent series for $D(1 - s)$ and the gamma and trigonometric factors. $\square$

For integers $q, k \geq 1$, define

$$T_{q,k}(X) = \sum_{n \leq X^q} \Lambda(n)(1 - n^{1/q}/X)^k, \quad K_k(s) = \frac{k!}{s(s+1) \cdots (s+k)}, \quad C_{q,k} = \frac{q!k!}{(q+k)!}.$$

The weight vanishes at its right endpoint.

Theorem 7.2 (A zero-free multiplier for compact windows). *For $\Re s > q$,*

$$(7.2) \quad \int_0^\infty T_{q,k}(X) X^{-s-1} dX = K_k(s) D(s/q).$$

For every fixed $q, k \geq 1$,

$$(7.3) \quad \text{RH} \iff T_{q,k}(X) = C_{q,k} X^q + O_{q,k}(X^{q/2}).$$

More precisely, for $0 < \eta < 1$,

$$(7.4) \quad T_{q,k}(X) = C_{q,k} X^q - q \sum_{\rho} m_{\rho} K_k(q\rho) X^{q\rho} - \log(2\pi) + O_{q,k,\eta}(X^{-\eta}),$$

where the zero sum converges absolutely for fixed $X > 0$. The infimum of exponents A for which the centered error is $O(X^A)$ is $q\Theta$.

Proof. For a fixed input n , substitute $u = n^{1/q}/X$ in its integral over $X \geq n^{1/q}$. The result is

$$n^{-s/q} \int_0^1 u^{s-1} (1-u)^k du = K_k(s) n^{-s/q}.$$

The beta integral follows by k integrations by parts. Absolute convergence permits summation, proving (7.2).

Mellin inversion is absolutely convergent on a line $c > q$, because $K_k(c+it) = O((1+|t|)^{-k-1})$ and $D(s/q)$ has an absolutely convergent Dirichlet series there. Shift to $\Re s = -\eta$. Lemma 7.1, applied with heights scaled by q , makes the horizontal integrals tend to zero. On the left line the integral is absolutely convergent and bounded by a constant times $X^{-\eta}$. The pole at q has residue $qK_k(q)X^q = C_{q,k}X^q$. A zero ρ gives residue $-qm_{\rho}K_k(q\rho)X^{q\rho}$. The pole of K_k at zero gives $D(0) = -\log(2\pi)$. The other rational-kernel poles and the trivial-zero poles lie outside this strip. The bound $N(T) = O(T \log T)$ and $K_k(q\rho) = O((1+|\Im \rho|)^{-k-1})$ prove absolute convergence of the zero sum, including $k = 1$.

Under RH every $X^{q\rho}$ has modulus $X^{q/2}$, proving the forward implication in (7.3). Conversely $T_{q,k}(X) = 0$ for $X \leq 1$, so initially in $\Re s > q$,

$$\int_1^\infty (T_{q,k}(X) - C_{q,k} X^q) X^{-s-1} dX = K_k(s) D(s/q) - \frac{C_{q,k}}{s-q}.$$

An error $O(X^A)$ continues the left side holomorphically to $\Re s > A$. A zero with $q\Re \rho > A$ would give a pole with nonzero residue on the right, since K_k has no zeros. It is therefore excluded. Reflection proves RH for $A = q/2$. Formula (7.4) also bounds the error by $O(X^{q\Theta})$; the same pole obstruction for each $A < q\Theta$ proves the infimum assertion. $\square$

A conjugate pair $\rho = \beta + i\gamma$, $\bar{\rho}$ contributes

$$-2q m_{\rho} X^{q\beta} \Re(K_k(q\rho) e^{iq\gamma \log X}).$$

Its frequency is $q\gamma$ in $\log X$. Zeros of a single unweighted rounded function F_q or C_q need not be these frequencies: (3.4) uses their difference to recover ζ exactly.

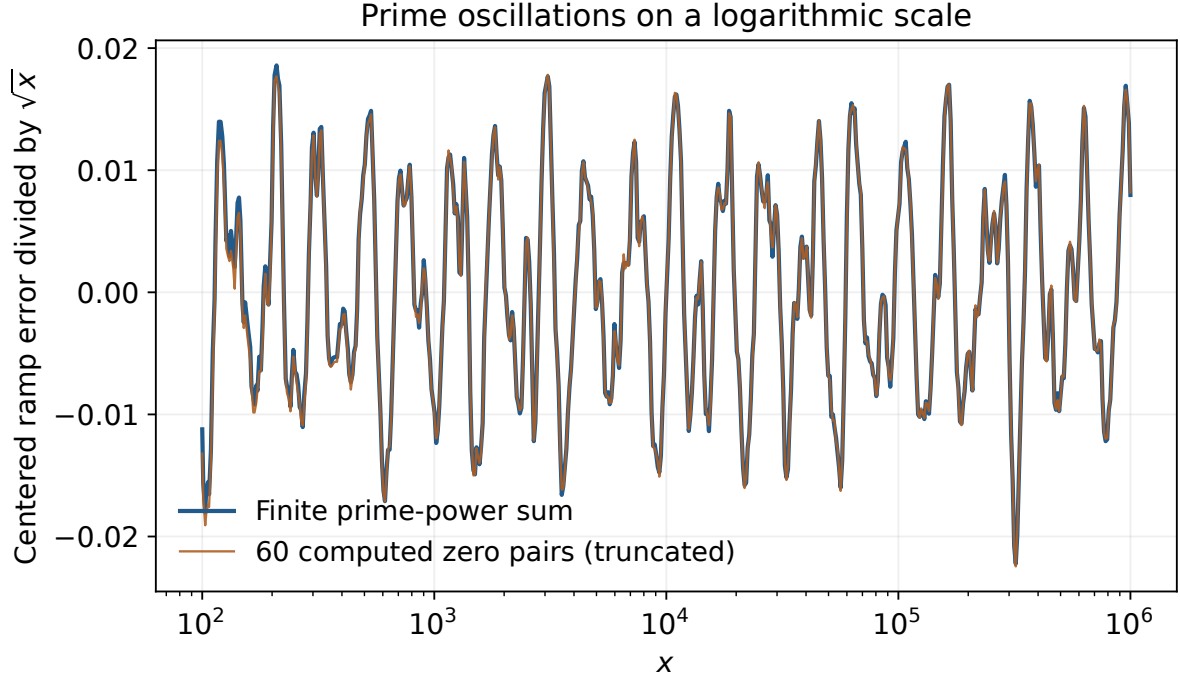


FIGURE 3. The centered $q = k = 1$ Mangoldt window and the contribution of 60 computed critical-line zero pairs. The centering adds $\log(2\pi)$ before division by $\sqrt{X}$. The omitted zero modes are not bounded by this finite comparison.

7.1. Literal rounding and its reference terms. For $w_{X,k}(t) = (1 - t/X)_+^k$, the literal floor transform is

$$\sum_n \Lambda(n) w_{X,k}(h_{F,q}(n)) = \sum_{j \geq 1} (\psi((j+1)^q - 1) - \psi(j^q - 1)) w_{X,k}(j).$$

This is a finite sum; its weights have the factorial representation in Proposition 3.1. The corresponding unweighted reference at an integer $X = J$ is

$$M_{q,k}^F(J) = \sum_{j=1}^{J-1} f_q(j) (1 - j/J)^k.$$

Expanding the finite polynomials, or using the power-sum identities of [1], gives

$$(7.5) \quad M_{2,1}^F(J) = J^2/3 + J/2 - 5/6, \quad M_{2,1}^C(J) = J^2/3 - J/2 + 1/6.$$

Indeed substitute $f_2(j) = 2j + 1$ or $c_2(j) = 2j - 1$, and use $\sum_{j < J} j = J(J-1)/2$ and $\sum_{j < J} j^2 = J(J-1)(2J-1)/6$. These deterministic linear terms must be separated from arithmetic oscillation.

Proposition 7.3 (Integer-node smoothing). *Let $k \geq L \geq 1$, $j_n = h_{F,q}(n)$ and $\delta_n = n^{1/q} - j_n$. Then*

$$T_{q,k,L}^{\text{int}}(X) = \sum_n \Lambda(n) \sum_{\nu < L} \ell_\nu(\delta_n) w_{X,k}(j_n + \nu)$$

satisfies $T_{q,k,L}^{\text{int}} - T_{q,k} = O_{q,k,L}(X^{q-L})$. Thus for $L \geq \lceil q/2 \rceil$ its error $O(X^{q/2})$ about $C_{q,k}X^q$ is equivalent to RH.

Proof. The L th derivative of $w_{X,k}$ is bounded almost everywhere by $C_{k,L}X^{-L}$. For $k = L$ the function has $L - 1$ absolutely continuous derivatives locally, with bounded L th weak derivative. The divided-difference remainder used in Lemma 5.3 therefore applies, either directly by its integral form or by smooth approximation, giving error $O_{k,L}(X^{-L})$ per input. Both the interpolant and the continuous window vanish when $j_n \geq X$. Hence only $n < (X+1)^q$

contributes. The Chebyshev estimate $\psi(y) = O(y)$ bounds their total mass by $O_q(X^q)$. This proves the error; Theorem 7.2 gives the equivalence. $\square$

Unconditionally Stieltjes integration by parts gives

$$(7.6) \quad T_{q,k}(X) - C_{q,k}X^q = \frac{k}{qX} \int_0^{X^q} (\psi(t) - t)t^{1/q-1}(1 - t^{1/q}/X)^{k-1} dt.$$

The PNT, $\psi(t) - t = o(t)$, proves the baseline $o(X^q)$. The stronger square-root estimate is an additional arithmetic assertion.

8. A POSITIVE KERNEL FROM THE CUBIC DIFFERENCE

Equation (2.6) suggests subtracting its explicit singular power. Define

$$(8.1) \quad A(t) = \frac{1}{e^t - 1}, \quad d(t) = \frac{1}{t^2} - \frac{e^{-t}}{(1 - e^{-t})^2}, \quad w(u) = d(u^2).$$

Then the regularized cubic heat difference is $-6d(t)$. The following properties are proved here from the elementary geometric sum and its public hyperbolic product.

Lemma 8.1 (Bounds and ordinary Mellin integral). *For $t > 0$,*

$$0 < d(t) \leq \min\{1/12, t^{-2}\}, \quad d(t) = 1/12 - t^2/240 + O(t^4) \quad (t \downarrow 0).$$

The ordinary convergent integral satisfies

$$(8.2) \quad \int_0^\infty d(t)t^{s-1} dt = -\Gamma(s)\zeta(s-1), \quad 0 < \Re s < 2.$$

Consequently

$$(8.3) \quad K_d(s) := \int_0^\infty w(u)u^{s-1} du = -\frac{1}{2}\Gamma(s/2)\zeta(s/2-1), \quad 0 < \Re s < 4.$$

Proof. Since $e^{-t}/(1 - e^{-t})^2 = 1/(4\sinh^2(t/2))$ and $2\sinh(t/2) > t$, positivity and $d(t) \leq t^{-2}$ follow. The classical partial fraction expansion of $\coth$ [3, §4.36] gives

$$A(t) - t^{-1} = -\frac{1}{2} + 2t \sum_{\ell \geq 1} \frac{1}{(2\pi\ell)^2 + t^2}.$$

Differentiate, normally on compact positive intervals, to obtain

$$d(t) = 2 \sum_{\ell \geq 1} \frac{(2\pi\ell)^2 - t^2}{((2\pi\ell)^2 + t^2)^2}.$$

Each term is at most $2/(2\pi\ell)^2$; summing proves $d \leq 1/12$. Expanding $t/(e^t - 1)$, or the displayed partial fractions near zero, gives the stated Taylor terms.

Let $h(t) = -A'(t) = \sum_{j \geq 1} j e^{-jt}$. In $\Re s > 2$ its Mellin integral is $\Gamma(s)\zeta(s-1)$ by absolute integration of the geometric series. Splitting at one and subtracting t^{-2} continues this expression to $\Re s > 0$ as

$$\int_0^1 (h(t) - t^{-2})t^{s-1} dt + \int_1^\infty h(t)t^{s-1} dt + \frac{1}{s-2}.$$

For $0 < \Re s < 2$ it is exactly $-\int_0^\infty d(t)t^{s-1} dt$, because $\int_1^\infty t^{s-3} dt = -1/(s-2)$. This proves (8.2) without a divergent-integral convention. Substitute $t = u^2$ for (8.3). $\square$

Define the positive traces

$$(8.4) \quad \mathcal{P}(X) = \sum_n \Lambda(n)w(n/X), \quad \mathcal{A}(X) = \sum_p \log p w(p/X), \quad \Pi_d(X) = \sum_p w(p/X).$$

They converge absolutely and locally uniformly, since $w(n/X) \leq X^4/n^4$.

Theorem 8.2 (Noncancellation and the error frontier). *Put*

$$c_d = K_d(1) = 0.18423437000566128596 \dots$$

For $1 < c < 4$,

$$(8.5) \quad \mathcal{P}(X) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_d(s) D(s) X^s ds.$$

For every nontrivial Riemann zero, $K_d(\rho) \neq 0$. Unconditionally,

$$\mathcal{P}(X) \sim c_d X, \quad \mathcal{A}(X) \sim c_d X, \quad \Pi_d(X) \sim c_d X / \log X.$$

Moreover,

$$(8.6) \quad \text{RH} \iff \mathcal{P}(X) - c_d X = O_\epsilon(X^{1/2+\epsilon}) \text{ for every } \epsilon > 0.$$

The infimum of b for which the error is $O_\epsilon(X^{b+\epsilon})$ for every $\epsilon > 0$ is Θ .

Proof. Mellin inversion in (8.3), followed by the absolutely convergent Dirichlet series for D , proves (8.5). Stirling's formula supplies exponential vertical decay $e^{-\pi|\Im s|/4}$ times a polynomial. At a nontrivial ρ , the argument $\rho/2 - 1$ has real part in $(-1, -1/2)$ and nonzero imaginary part. Zeta has no zero there: all its left-half-plane zeros are negative even integers. Gamma has no zeros or poles at $\rho/2$. Thus $K_d(\rho) \neq 0$ and $K_d D$ has residue $-m_\rho K_d(\rho)$.

The PNT gives $\psi(Xu)/X \rightarrow u$ on fixed positive compact intervals. The bounds $w(u) \leq 1/12$ near zero and $w(u) \leq u^{-4}$ at infinity, together with $\psi(y) = O(y)$, bound the two omitted tails by $O(a)$ and $O(B^{-3})$ after division by X , on $u < a$ and $u > B$. Passing to the Stieltjes limit on $[a, B]$ and then taking $a \downarrow 0$, $B \uparrow \infty$ gives $\mathcal{P}(X)/X \rightarrow \int_0^\infty w = c_d$. The same proof with θ gives $\mathcal{A}$. For Π_d , use the PNT for π , normalizing by $X/\log X$. Its large- u tail is controlled by the PNT upper bound, and its range $u < X^{-1/2}$ contributes $O(X^{-1/2} \log X)$ after normalization. The remaining small- u range is $O(a)$ because $\log(Xu) \geq \frac{1}{2} \log X$. This proves all three main terms.

At zero $\mathcal{P}(X) = O(X^4)$. Hence initially for $1 < \Re s < 4$,

$$(8.7) \quad \int_0^1 \mathcal{P}(X) X^{-s-1} dX + \int_1^\infty (\mathcal{P}(X) - c_d X) X^{-s-1} dX = K_d(s) D(s) - \frac{c_d}{s-1}.$$

An error of exponent $b+\epsilon$ for every $\epsilon > 0$ continues the left side to $b < \Re s < 4$. Noncancellation excludes a zero with $\Re \rho > b$, proving $\Theta \leq b$. Conversely, if $\Theta < 1$, shift (8.5) to any line $\Theta < b < 1$, crossing only the pole at one. Lemma 7.1 and exponential gamma decay justify the shift and bound the remaining integral by $O_b(X^b)$. For $\Theta = 1$, the unconditional $O(X)$ bound gives the endpoint frontier. Reflection of zeros proves (8.6). $\square$

Proposition 8.3 (An absolutely convergent explicit formula). *For $2 < \eta < 4$ and $X \geq 1$,*

$$(8.8) \quad \mathcal{P}(X) = c_d X - \sum_\rho m_\rho K_d(\rho) X^\rho - \frac{\log(2\pi)}{12} - K_d(-2) X^{-2} + O_\eta(X^{-\eta}),$$

where $K_d(-2) = \zeta'(-2)/2$. The zero sum is absolutely convergent.

Proof. Shift (8.5) to $\Re s = -\eta$, using Lemma 7.1. Exponential decay makes both the vertical integral and the zero sum absolutely convergent. The pole at one gives $c_d X$, and the nontrivial zeros give the displayed residues. At zero, K_d has residue $1/12$ and $D(0) = -\log(2\pi)$. At -2 the pole of $\Gamma(s/2)$ cancels the simple zero of $\zeta(s/2 - 1)$, leaving $K_d(-2) = \zeta'(-2)/2$; the simple pole of D still contributes $-K_d(-2) X^{-2}$. The small- u expansion of $w(u) = 1/12 - u^4/240 + O(u^8)$ shows that the next kernel pole is at -4 , outside the strip. These exhaust the crossed poles, and the remaining integral is $O_\eta(X^{-\eta})$. $\square$

9. PRIME-POWER BIAS AND COMPLETE MOMENTS

Theorem 9.1 (The square-power bias). *Unconditionally,*

$$(9.1) \quad \mathcal{P}(X) - \mathcal{A}(X) \sim B_d \sqrt{X}, \quad B_d = \frac{1}{2} K_d(1/2) = 0.12113414178988663741 \dots$$

Powers of exponent at least three contribute $O(X^{1/3} \log(2X))$.

Proof. The square contribution is $\sum_p \log p d(p^4/X^2)$. By the PNT for θ and the substitution $p = \sqrt{X} u$, its asymptotic is

$$\sqrt{X} \int_0^\infty d(u^4) du = \frac{1}{2} K_d(1/2) \sqrt{X}.$$

The endpoint bounds in Lemma 8.1 justify the Stieltjes limit as above. For the remaining powers their cumulative mass through y is bounded by

$$\sum_{3 \leq k \leq \log_2 y} \theta(y^{1/k}) = O(y^{1/3} \log(2y)).$$

Use $w \leq 1/12$ below $p^k = X$, and $w \leq X^4/(p^k)^4$ above it. Integration by parts with the displayed cumulative bound gives the asserted $O(X^{1/3} \log(2X))$, which is $o(\sqrt{X})$. $\square$

The prime-only trace therefore contains a deterministic term of the same order as critical-line oscillations. Its removal is part of interpreting the error, not evidence of a new zero. Both traces are exactly related by (3.2); the absolutely convergent infinite version is justified by summing the prime-power terms against $w(n/X) \ll X^4 n^{-4}$.

Let

$$M_{\Lambda,4} = D(4) = \sum_n \frac{\Lambda(n)}{n^4}, \quad M_{p,4} = \sum_p \frac{\log p}{p^4} = \sum_{k \geq 1} \mu(k) D(4k).$$

The last identity is absolutely convergent Möbius inversion; the same divisor sum as in (3.2) proves it. Equation (3.4) expresses $D(4)$ through the original rounded pair and its derivatives.

Theorem 9.2 (Complete fourth-moment acceleration). *For $N \geq \max\{2, X\}$ put*

$$\mathcal{P}_N^+(X) = \sum_{n \leq N} \Lambda(n) w(n/X) + X^4 \left(D(4) - \sum_{n \leq N} \frac{\Lambda(n)}{n^4} \right).$$

Then

$$(9.2) \quad 0 \leq \mathcal{P}_N^+(X) - \mathcal{P}(X) \leq \frac{X^2(\log N + 1)}{2N} \frac{e^{-(N/X)^2}}{(1 - e^{-(N/X)^2})^2}.$$

For $\mathcal{A}$ replace $D(4)$ by $M_{p,4}$ and restrict the finite sums to primes. Its moment truncation after $k = K$ has error at most

$$(9.3) \quad \left| M_{p,4} - \sum_{k \leq K} \mu(k) D(4k) \right| \leq \frac{16}{15} D(4) 2^{-4K}.$$

Proof. For every omitted input,

$$w(n/X) = X^4 n^{-4} - \frac{e^{-(n/X)^2}}{(1 - e^{-(n/X)^2})^2}.$$

The complete moment sums the first term exactly; the subtracted remainder is nonnegative. Bound its denominator by the value at N , use $\Lambda(n) \leq \log n$, and compare the decreasing function $\log t e^{-t^2/X^2}$ with its integral from N . Since

$$\log t \leq \log N + t/N, \quad \int_N^\infty e^{-t^2/X^2} dt \leq \frac{X^2}{2N} e^{-N^2/X^2}, \quad \int_N^\infty t e^{-t^2/X^2} dt = \frac{X^2}{2} e^{-N^2/X^2},$$

the resulting bound is (9.2). The prime-restricted error is smaller than the same integer majorant. Finally $D(4k) \leq D(4)2^{4-4k}$ by its positive Dirichlet coefficients supported on $n \geq 2$; summing the geometric tail proves (9.3). $\square$

Without acceleration, the elementary estimate is only

$$0 \leq \mathcal{P}(X) - \sum_{n \leq N} \Lambda(n)w(n/X) \leq \frac{X^4}{N^3} \left(\frac{\log N}{3} + \frac{1}{9} \right),$$

obtained by integrating $\log t/t^4$. Thus the complete moment changes the analytic truncation error substantially. The subtraction of the finite fourth moment from $D(4)$ requires adequate numerical precision; its two terms have size $O(1)$ whereas their difference has size $O(N^{-3})$. This acceleration does not estimate the centered arithmetic error in (8.6).

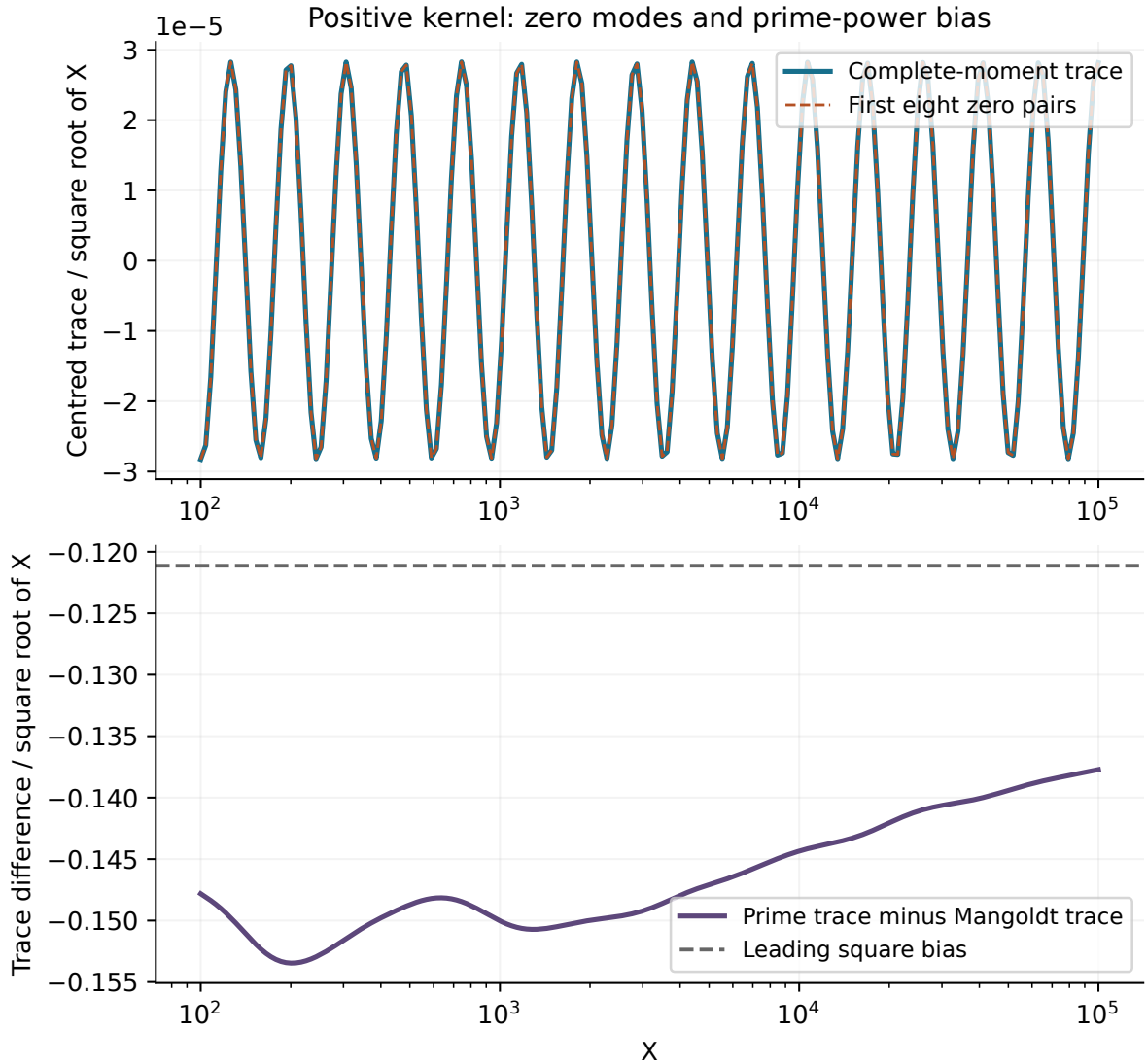


FIGURE 4. Positive-kernel traces on $100 \leq X \leq 10^5$. Above: the Mangoldt trace centered by the main, constant and first trivial-zero terms in (8.8), divided by $\sqrt{X}$, compared with eight known zero pairs. Below: $(\mathcal{A} - \mathcal{P})/\sqrt{X}$ and the leading square bias $-B_d$. Complete fourth-moment tails are included. The finite comparison is not an interval-certified complete zero census.

10. CLASSICAL BLOCK DUALITY AND A PRIME-SCALE BOUND

The Hilbert-space machinery in this section belongs to the Nyman–Beurling literature. We retain the proofs to make the normalization and the finite prime-only bound explicit. The dual family below is Vasyunin’s system in reciprocal coordinates.

A second exact level decomposition concerns reciprocal integer scales. For $n \geq 2$ and $0 < x \leq 1$, define

$$g_n(x) = \frac{1}{n} \left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{nx} \right\rfloor.$$

On $I_j = (1/(j+1), 1/j]$ it equals $(j \bmod n)/n$ almost everywhere, and $|I_j| = 1/[j(j+1)]$. In particular $0 \leq g_n < 1$ almost everywhere.

Proposition 10.1 (Mellin and Gram formulas). *For $\Re s > 0$,*

$$(10.1) \quad \int_0^1 g_n(x) x^{s-1} dx = \frac{n^{-1} - n^{-s}}{s} \zeta(s),$$

with removable value $\log n/n$ at $s = 1$. For $L = \text{lcm}(m, n)$ the Gram matrix entries are

$$(10.2) \quad G_{mn} = \sum_{j \geq 1} \frac{(j \bmod m)(j \bmod n)}{mn j(j+1)} \\ = \frac{1}{mnL} \sum_{r=1}^{L-1} (r \bmod m)(r \bmod n) \left[\Psi\left(\frac{r+1}{L}\right) - \Psi\left(\frac{r}{L}\right) \right].$$

Every finite such matrix for distinct scales is positive definite. For a finite set S , the minimum squared error in approximating one is

$$(10.3) \quad d_S^2 = 1 - b^T G_S^{-1} b, \quad b_n = \log n/n.$$

Proof. For $\Re s > 1$, expand $[a/x] = \sum_{k \geq 1} \mathbf{1}_{x \leq a/k}$ for $0 < a \leq 1$ and integrate. Its transform is $a^s \zeta(s)/s$. Subtract the two floor transforms. The bounded difference is holomorphic on $\Re s > 0$, so analytic continuation proves (10.1); expansion at one gives its mean.

Integration on the intervals I_j gives the absolutely convergent series (10.2). Split it into residues modulo L . The classical difference series for the digamma function gives

$$\sum_{k \geq 0} \left(\frac{1}{kL+r} - \frac{1}{kL+r+1} \right) = \frac{1}{L} \left[\Psi\left(\frac{r+1}{L}\right) - \Psi\left(\frac{r}{L}\right) \right],$$

proving the second formula. The term $r = L$ vanishes.

For independence, a finite relation $\sum c_n g_n = 0$ would give $\sum c_n (n^{-1} - n^{-s}) = 0$ for real $s > 1$. Taking $s \rightarrow \infty$ first removes its constant. Then multiplying successively by the smallest remaining base to the power s isolates its coefficient, forcing all coefficients to vanish. Thus the Gram matrix is positive definite. Completing the quadratic norm of $1 - \sum c_n g_n$ proves (10.3). $\square$

For an off-line zero $\rho = \beta + it$ with $\beta > 1/2$, equation (10.1) and Cauchy–Schwarz also give the classical obstruction

$$\left\| 1 - \sum_{n \in S} c_n g_n \right\|_2^2 \geq \frac{2\beta - 1}{|\rho|^2}.$$

Indeed the Mellin integral of the residual at ρ is $1/\rho$, while $\int_0^1 |x^{\rho-1}|^2 dx = (2\beta - 1)^{-1}$. The next obstruction instead comes from a missing scale, without assuming an off-line zero.

10.1. Identification with Vasyunin's dual functions. For an integer $m \geq 2$ set $A_m(j) = \mu(m/j)$ when $j \mid m$ and zero otherwise, and $a_m(j) = A_m(j) - A_m(j+1)$, with $A_m(j) = 0$ for $j > m$. Define v_m on I_j , $j \leq m$, by $v_m(x) = j(j+1)a_m(j)$, and zero on $0 < x \leq 1/(m+1)$.

Here is the precise identification with [9], using the sign convention of [10, §2]. Put $e_n(t) = \{t/n\}$ and

$$\begin{aligned}\varphi_d(t) &= d(d-1)\mathbf{1}_{[d-1,d)}(t) - d(d+1)\mathbf{1}_{[d,d+1)}(t), \\ F_m^V(t) &= \sum_{d \mid m} \mu(m/d) \varphi_d(t).\end{aligned}$$

On $[j, j+1)$ the latter function has value $j(j+1)(A_m(j+1) - A_m(j))$. Therefore

$$(10.4) \quad v_m(x) = -F_m^V(1/x), \quad g_n(x) = e_n(1/x) - e_1(1/x)/n.$$

Substitution $t = 1/x$ gives the unitary change of variables

$$\int_0^1 |F(1/x)|^2 dx = \int_1^\infty |F(t)|^2 \frac{dt}{t^2}.$$

Thus the next duality identity is exactly the classical one after this change of variable and sign. The finite-block proof also records its norm in the coordinates used for the prime-scale certificate.

Theorem 10.2 (Vasyunin duality in reciprocal blocks). *For every $n \geq 2$,*

$$(10.5) \quad \langle 1, v_m \rangle = \mu(m), \quad \langle g_n, v_m \rangle = -\mathbf{1}_{n=m}, \quad \|v_m\|_2^2 = \sum_{j=1}^m j(j+1)a_m(j)^2.$$

If a scale set omits a squarefree m , every approximant in its closed span has squared target error at least $1/\|v_m\|_2^2 > 0$.

Proof. The mean and norm follow by integration over I_j . Summation by parts gives

$$\sum_j a_m(j)j = \sum_j A_m(j) = \sum_{d \mid m} \mu(d) = 0,$$

and

$$\sum_j a_m(j) \lfloor j/n \rfloor = \sum_{n \mid j} A_m(j) = \begin{cases} \sum_{d \mid m/n} \mu(d), & n \mid m, \\ 0, & n \nmid m. \end{cases}$$

This is one exactly when $n = m$ and zero otherwise. Insert $g_n = j/n - \lfloor j/n \rfloor$ to prove the middle identity. If m is omitted, v_m is orthogonal to the closed span, so $|\mu(m)| \leq \|1 - f\|_2 \|v_m\|_2$. For squarefree m , $|\mu(m)| = 1$. $\square$

10.2. Coefficient normalization and the prime-only certificate. For a finite combination $f = \sum_n c_n g_n$, extend its coefficients by zero at absent scales. Equation (10.5) gives

$$(10.6) \quad |c_m + \mu(m)| = |\langle 1 - f, v_m \rangle| \leq \|v_m\|_2 \|1 - f\|_2.$$

Consequently any sequence of such combinations converging to one must have $c_m \rightarrow -\mu(m)$ for each fixed m . This recovers the classical coefficient-uniqueness obstruction: omitting a squarefree scale prevents convergence.

To compare directly with Gallardo-Gutiérrez and Seco [11, Introduction], define $r_n(j) = n\{j/n\} = j \bmod n$. On I_j , $g_n = r_n(j)/n$, so coefficients in $\sum_n a_n r_n$ and $\sum_n c_n g_n$ obey $a_n = c_n/n$. The sequence-space and block norms are equivalent: for $F_u|_{I_j} = u_j$,

$$(10.7) \quad \frac{1}{2} \sum_{j \geq 1} \frac{|u_j|^2}{j^2} \leq \|F_u\|_2^2 = \sum_{j \geq 1} \frac{|u_j|^2}{j(j+1)} \leq \sum_{j \geq 1} \frac{|u_j|^2}{j^2}.$$

It follows that the limit $a_m \rightarrow -\mu(m)/m$ recorded there is precisely $c_m \rightarrow -\mu(m)$ here. This comparison requires no new completeness or uniqueness theorem. The following computation

uses two of the classical dual vectors to obtain a quantitative bound for the prime-restricted scale set.

Corollary 10.3 (Prime-only obstruction).

$$(10.8) \quad \inf_{f \in \text{span}\{g_p : p \in \mathbb{P}\}} \|1 - f\|_2^2 \geq \frac{5}{428} = 0.01168224299065 \dots$$

Proof. The coefficient vectors for v_6 and v_4 , padded to six blocks, are $(2, 0, -1, 0, -1, 1)$ and $(1, -1, -1, 1, 0, 0)$. Both functions are orthogonal to every prime scale. The combination $v = 5v_6 - 2v_4$ has block coefficients $(8, 2, -3, -2, -5, 5)$, mean five and squared norm

$$2 \cdot 8^2 + 6 \cdot 2^2 + 12 \cdot 3^2 + 20 \cdot 2^2 + 30 \cdot 5^2 + 42 \cdot 5^2 = 2140.$$

Cauchy–Schwarz gives $25/2140 = 5/428$. In this two-function family the combination is optimal, since $\|v_4\|^2 = 40$, $\|v_6\|^2 = 92$ and $\langle v_4, v_6 \rangle = 16$. We do not assert that it gives the exact distance to the full prime-scale closure. $\square$

The justified restriction is squarefree support. The public integer-scale Nyman–Beurling criterion [8] states that RH is equivalent to approximating $\chi = \mathbf{1}_{(0,1)}$ in $L^2(0, \infty)$ by finite spans of $\rho_n(x) = \{1/(nx)\}$. Its proof under RH supplies Möbius-smoothed approximants, whose coefficients vanish on nonsquarefree indices. If $F = \sum a_n \rho_n$ approximates χ , its tail on $x > 1$ is A/x , $A = \sum a_n/n$, so $|A| \leq \|F - \chi\|_2$. Balancing gives $F - A\rho_1 = \sum_{n \geq 2} a_n g_n$, extended by zero on $x > 1$, and changes the norm by at most $|A|\|\rho_1\|_2$. Therefore

$$(10.9) \quad \text{RH} \iff 1 \in \overline{\text{span}\{g_n : n \geq 2, n \text{ squarefree}\}}.$$

The converse uses the same public criterion, since $g_n = \rho_n - \rho_1/n$. The necessity of every squarefree scale is the coefficient-uniqueness consequence just compared with the literature. Corollary 10.3 quantifies one obstruction for the prime-restricted closure.

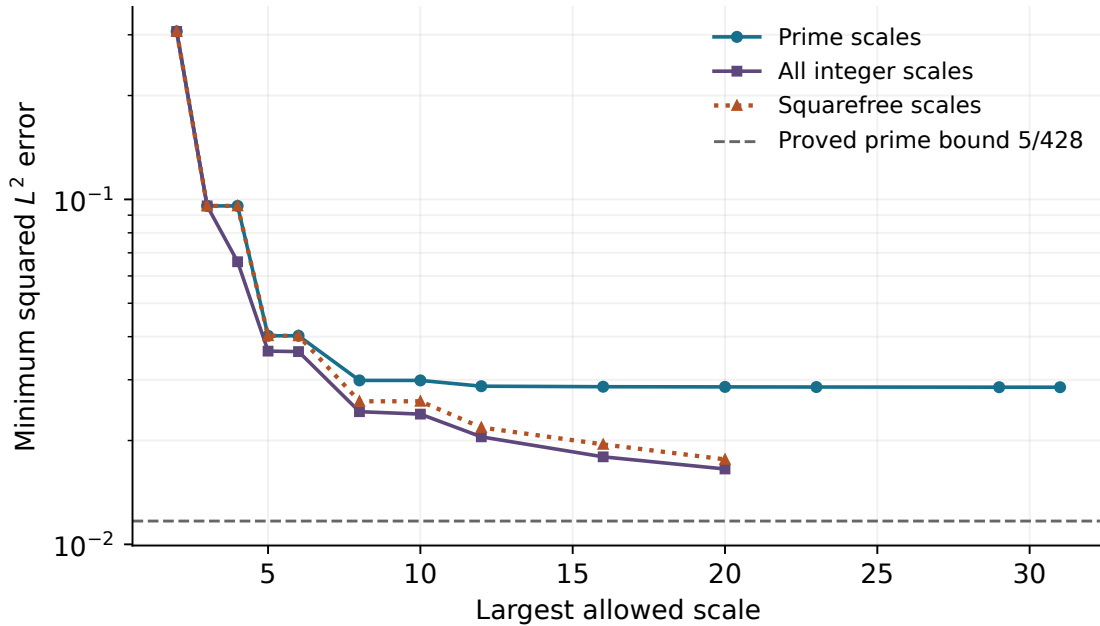


FIGURE 5. Finite minimum squared errors from the exact Gram formula, for integer, squarefree and prime-only scale sets. The dashed line is the proved lower bound (10.8). Finite improvement in the unrestricted or squarefree curves does not prove convergence.

11. COMPUTATION AND THE REMAINING ARITHMETIC ESTIMATES

The ancillary programs generate their own prime tables. They use integer or rational arithmetic for finite fibre identities, prime-power extraction, the scaled-hit classification, adjacent-fraction certificates, occupation coefficients and block duality. Floating or arbitrary-precision calculations check special-function formulas, interpolation tails, complete moments and Gram matrices. Their mathematical truncation bounds are proved above; floating arithmetic errors are not certified by interval arithmetic.

The main sieve cutoff is 2 000 000, where $\pi(2\,000\,000) = 148\,933$. The scale certificate of Example 4.3 is checked using integer squaring, determinants and containment comparisons. The occupation recurrence is checked against independent direct multiplication of the factors through degree 30, for $q = 2, 3$ and both rounding directions. The block dual identities are checked with rational arithmetic, and the Gram implementation is compared with direct interval integration with a tail bound. A further 236 exact cases compare the two definitions in (10.4), the sequence normalization, the norm weights and the coefficient functionals. All 16 945 supplied finite checks pass.

For the positive trace, complete fourth moments use 55 decimal digits; the prime fourth moment uses 40 Möbius terms. The omitted prime moment in (9.3) is below 4.647×10^{-50} before multiplication by X^4 . The finite Mangoldt fourth-moment tail after the sieve cutoff is approximately $4.1663400363 \times 10^{-20}$. Near zero the implementation evaluates d by its convergent Bernoulli series to avoid subtracting two terms of size t^{-2} . Over the plotted range the centered trace and eight-zero-pair model differ numerically by less than 2.4×10^{-11} . This finite agreement is not a completeness proof for a zero list. Noncancellation for every nontrivial zero follows from Theorem 8.2, not from the plot.

TABLE 1. Selected finite minimum squared errors, computed at 55-digit working precision from (10.2). Displayed decimals are descriptive, not interval-certified enclosures.

Largest scale	All integers	Squarefree scales	Prime scales
4	0.06595169	0.09574778	0.09574778
8	0.02424453	0.02598729	0.02987100
12	0.02049815	0.02181608	0.02873926
20	0.01653825	0.01763282	0.02860472
31	—	—	0.02854502

The finite error of the first compact window at 10^6 is $T_{1,1}(10^6) - 500000 = 6.136687\dots$, and $\pi(10^6) = 78\,498$. At $t = 0.012$, $q = 2$ and scale two, the ceiling discrepancy in (6.3) is approximately -58.4737192 , compared with leading prediction -61.5175209 ; the total omitted prime tails are bounded by 0.005328 using the integral majorant for $\sum_{n>N} e^{-t\sqrt{\beta n}}$. These examples illustrate different proved formulas without establishing their unknown uniform remainders.

The global target for the positive kernel remains proving the error estimate in (8.6) without assuming RH; the compact target is (7.3). Certified scales remove uncertainty in rounded labels, complete moments remove numerical tails, and the block dual functions exclude certain restricted approximations. None of these steps supplies the missing arithmetic cancellation. The square-block Legendre target and asymptotic prime-tuple estimates likewise require additional input. The constructions above give explicit representations and obstructions with these proof obligations visible.

APPENDIX A. PRIME PRODUCTS, DETERMINANTS AND MOMENT CANCELLATION

Since $h_{F,q}(p) \sim p^{1/q}$, for fixed $y \geq 0$ and $m > 0$,

$$(A.1) \quad \sum_p (h_{F,q}(p) + y)^{-m} < \infty \iff m > q.$$

For $m = q$ this follows from the prime harmonic divergence; for $m < q$ the terms eventually dominate a constant times $1/p$. For $m > q$ the full integer comparison converges.

Proposition A.1 (The critical normalization). *For fixed $t > 0$, $y \geq 0$ and $q \geq 2$,*

$$\prod_{p \leq Y} \left(1 + \frac{t}{(h_{F,q}(p) + y)^q} \right) \sim C_{q,t,y} (\log Y)^t,$$

where $C_{q,t,y} > 0$ and

$$\log C_{q,t,y} = tB_1 + \sum_p \left[\log \left(1 + \frac{t}{(h_{F,q}(p) + y)^q} \right) - \frac{t}{p} \right].$$

Here B_1 is the constant in the classical Mertens formula $\sum_{p \leq Y} 1/p = \log \log Y + B_1 + o(1)$.

Proof. The bracket is $O(p^{-1-1/q} + p^{-2})$, hence summable. Add Mertens' asymptotic [3, §27.11] and exponentiate. $\square$

Moment cancellation gives a general convergent family. Fix an integer $L \geq 1$. Let finitely many shifts $y_\nu \geq 0$ and real weights λ_ν satisfy $\sum_\nu \lambda_\nu y_\nu^r = 0$ for $0 \leq r < L$, while $M_L = \sum_\nu \lambda_\nu y_\nu^L \neq 0$. Taylor expansion yields

$$\log b(j) := \sum_\nu \lambda_\nu \log(j + y_\nu) = cj^{-L} + O(j^{-L-1}), \quad c = (-1)^{L+1} M_L / L.$$

Its eventual sign is fixed. By (A.1), $\prod_p b(h_{F,q}(p))$ has a finite nonzero ordinary limit exactly when $L > q$. At $L = q$ the logarithm is $c \log \log Y + O(1)$ through $p \leq Y$. Pairing floor and ceiling improves the order because

$$\log b(h_{F,q}(p)) - \log b(h_{C,q}(p)) = Lc h_{F,q}(p)^{-L-1} + O(h_{F,q}(p)^{-L-2}).$$

Its convergence threshold is $L + 1 > q$. This proof uses only Taylor expansion, the exact no-hit identity, and the prime harmonic comparison.

For example, with $q = 2$ the relative product

$$R = \prod_p \frac{(h_{F,q}(p) + 1)^3 (h_{F,q}(p) + 3)}{h_{F,q}(p) (h_{F,q}(p) + 2)^3}$$

converges, and its cutoff R_Y for integer $Y \geq 4$ satisfies

$$(A.2) \quad 0 \leq \log(R/R_Y) \leq 32Y^{-1/2}.$$

Indeed its logarithmic factor is the third finite difference of $\log j$, equal to the integral of $2/(j+u+v+w)^3$ over $[0, 1]^3$. It lies in $(0, 2/j^3]$. Use $h_{F,q}(p) \geq \sqrt{p}/2$ and $\sum_{n > Y} n^{-3/2} \leq 2/\sqrt{Y}$.

A.1. Canonical and finite spectral products. For $q \geq 2$ define the standard factors

$$E_g(w) = (1 - w) \exp \left(\sum_{\ell=1}^g w^\ell / \ell \right).$$

Then

$$(A.3) \quad \mathcal{B}_q(z) = \prod_p E_q(-z/h_{F,q}(p))$$

is entire, with a zero at $-j$ of multiplicity $b_{q,j}$, and genus q is minimal for these standard factors. To prove this, on each compact set expand the logarithmic tail as $O(h_{F,q}(p)^{-q-1})$ and use (A.1). At genus $q - 1$ the first omitted Taylor term has fixed nonzero sign at a fixed

positive real test argument and its sum diverges like $\sum_p 1/p$. The tail is zero-free where its factors are, so the finite factors give exactly the asserted zeros.

The paired meromorphic product

$$\prod_p \frac{E_{q-1}(-z/h_{F,q}(p))}{E_{q-1}(-z/h_{C,q}(p))}$$

converges normally off its candidate poles. Its logarithmic tail is $O_K(h_{F,q}(p)^{-q-1})$, obtained by subtracting the first surviving power at j and $j+1$. Its signed multiplicity at $-j$ is $b_{q,j} - b_{q,j-1}$, with $b_{q,0} = 0$. Genus $q-2$ instead leaves a fixed nonzero multiple of j^{-q} at a positive test argument, so $q-1$ is minimal for this paired construction. These are zeros in the spectral shift variable z , not zeros in the Dirichlet exponent s .

There is also a direct finite determinant interpretation, with no infinite Gaussian normalization. On one coordinate for every prime $p \leq N$, let L_N be diagonal with entry $h_{F,q}(p)$. Then

$$\det(L_N + zI) = \prod_{p \leq N} (h_{F,q}(p) + z) = \prod_j (j + z)^{b_{q,j}(N)}.$$

For real $z > -1$, multiplication of the one-dimensional Gaussian integrals gives

$$\int_{\mathbb{R}^{\pi(N)}} e^{-\langle x, (L_N + zI)x \rangle / 2} dx = (2\pi)^{\pi(N)/2} \det(L_N + zI)^{-1/2}.$$

The determinant derivative gives the finite resolvent trace. In the canonical limit the convergent higher derivative is

$$\frac{d^{q+1}}{dz^{q+1}} \log \mathcal{B}_q(z) = (-1)^q q! \sum_p (h_{F,q}(p) + z)^{-q-1} \quad (\Re z > -1).$$

Normal convergence justifies differentiation, and the q subtraction polynomials vanish under these $q+1$ derivatives. Thus the normalized products and the resolvent observables are derived from the same weighted level measure.

APPENDIX B. OCCUPATION PRODUCTS AND RECOVERY OF THE PRIME FIBRES

For $q \geq 2$ and $X = F, C$, let $b_{q,j}^X$ be the unit-scale prime fibre counts. The colored-partition product [13] specializes to

$$Z_{q,X}(z) = \prod_{j \geq 1} (1 - z^j)^{-b_{q,j}^X} = \prod_p (1 - z^{h_{X,q}(p)})^{-1} \quad (|z| < 1).$$

Each prime labels one mode or color of energy $h_{X,q}(p)$.

Theorem B.1 (Occupation transform, inversion and boundary). *The product is analytic and nonzero on the disk, and*

$$(B.1) \quad \log Z_{q,X}(z) = \sum_{r \geq 1} \frac{G_{X,q}(z^r)}{r}.$$

Write $Z_{q,X}(z) = \sum_{N \geq 0} a_N z^N$ and $z Z'_{q,X} / Z_{q,X} = \sum_{N \geq 1} B_N z^N$. Then

$$(B.2) \quad B_N = \sum_{j|N} j b_{q,j}^X, \quad N a_N = \sum_{k=1}^N B_k a_{N-k}, \quad a_0 = 1, \quad b_{q,N}^X = \frac{1}{N} \sum_{j|N} \mu(N/j) B_j.$$

Moreover

$$(B.3) \quad \log Z_{q,X}(e^{-t}) \sim \Gamma(q) \zeta(q+1) \frac{t^{-q}}{\log(1/t)}.$$

The integer-coefficient Taylor series has radius one and the unit circle as a natural boundary.

Proof. The logarithmic series is normally absolutely convergent on compact subdisks because $b_{q,j}^X$ has polynomial growth. Expand $-\log(1 - z^j) = \sum_{r \geq 1} z^{jr}/r$ to prove (B.1), then differentiate and multiply by Z to obtain the first two formulas of (B.2). Divisor Möbius inversion gives the last. The product factors show that a_N are nonnegative integers.

By Theorem 6.1, $G_{X,q}(e^{-u}) \sim \Gamma(q)u^{-q}/\log(1/u)$. On the range $r \leq t^{-1/2}$ the summands in (B.1), multiplied by $t^q \log(1/t)$, have a uniform majorant Cr^{-q-1} . On all $u > 0$ the quantity $u^q G_{X,q}(e^{-u})$ is bounded, by its finite limit at zero and exponential decay at infinity. The remaining $r > t^{-1/2}$ tail is therefore $O(t^{q/2} \log(1/t))$. Dominated convergence on the first range gives $\Gamma(q) \sum_r r^{-q-1}$, proving (B.3).

The product diverges at one, so its analytic Taylor series has radius exactly one. A rational function has logarithmic radial growth at most $O(\log(1/t))$ there, incompatible with (B.3). Pólya–Carlson proves the natural boundary. $\square$

The first floor square-root coefficients are

$$(a_0, \dots, a_{10}) = (1, 2, 5, 10, 21, 38, 72, 125, 220, 368, 618).$$

They determine the original block counts through (B.2). This is a reversible encoding; it does not recover positions of primes inside a block without finer data. Likewise $G_{C,q} = zG_{F,q}$ does not imply $Z_{q,C} = zZ_{q,F}$: instead $\log Z_{q,C}(z) = \sum_{r \geq 1} z^r G_{F,q}(z^r)/r$.

For comparison with Dirichlet transforms, absolute Mellin inversion gives

$$\log Z_{q,F}(e^{-t}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s+1)Q_{F,q}(s)t^{-s} ds, \quad c > q.$$

The factor $\zeta(s+1)$ is zero-free on $\Re s > 0$, but $Q_{F,q}$ has prime-zeta logarithmic singularities after continuation. A coefficient asymptotic based on a simple meromorphic pole hypothesis would require additional justification. None is assumed here.

APPENDIX C. RESIDUE FILTERS AND LOCAL PRIME PATTERNS

For a Dirichlet character χ modulo d , the same Euler-logarithm argument as in (3.5) gives

$$\sum_p \chi(p)p^{-s} = \sum_{k \geq 1} \frac{\mu(k)}{k} \log L(k, \chi^k), \quad \Re s > 1.$$

At exponent m its coefficient is $\chi(p)^m m^{-1} \sum_{k|m} \mu(k)$; primes dividing the modulus are absent consistently. Character orthogonality, for $(u, d) = 1$, gives

$$\pi(Y; d, u) = \frac{1}{\varphi(d)} \sum_{\chi \bmod d} \overline{\chi(u)} \sum_{p \leq Y} \chi(p).$$

Proposition 2.2 then groups the input weights $\mathbf{1}_{n \in \mathbb{P}} \chi(n)$ into rounded fibres. Bounded periodic prefixes on all integers are not bounds for these character-weighted prime prefixes.

For an exact root-block sieve take $q, j \geq 2$, $L = j^q$, $U = (j+1)^q$, $z = \lfloor \sqrt{U} \rfloor$ and $P_z = \prod_{p \leq z} p$. Since $L \geq z$,

$$(C.1) \quad b_{q,j} = \sum_{d|P_z} \mu(d)(\lfloor U/d \rfloor - \lfloor L/d \rfloor).$$

Indeed inclusion–exclusion retains exactly the integers in $(L, U]$ with no prime divisor at most z . Every composite has such a divisor, and all primes in that interval exceed z . The perfect-power endpoint is nonprime. Small intervals with $j = 1$ need the small-prime addback and are not covered by this simplified form.

For a finite shift set $\mathcal{H} = \{h_1, \dots, h_m\}$, let $\Omega_p = \{-h_\nu \bmod p\}$ be a *set*, so repeated classes occur only once. For squarefree $d \mid P_z$, let $\Omega(d)$ be its CRT combinations, with $\Omega(1) = \{0\}$.

The exact count of integers avoiding these forbidden classes is

$$(C.2) \quad \sum_{d|P_z} \mu(d) \sum_{r \in \Omega(d)} \left(\left\lfloor \frac{U-r}{d} \right\rfloor - \left\lfloor \frac{L-r}{d} \right\rfloor \right).$$

Expand $\prod_{p \leq z} (1 - \mathbf{1}_{n \bmod p \in \Omega_p})$ and use the Chinese remainder theorem to prove this formula. If all $n + h_\nu > z$ on the interval and $z \geq \sqrt{U} + \max h_\nu$, the survivors are exactly prime tuples. If $|\Omega_p| = p$ for some p , every sufficiently large candidate tuple has a composite divisible by p . Otherwise the normalized local factors form the convergent positive singular series

$$\prod_p \frac{1 - |\Omega_p|/p}{(1 - 1/p)^m}.$$

For large p the shifts are distinct modulo p and its factor is $1 + O_{\mathcal{H}}(p^{-2})$, proving convergence. Its role in an asymptotic prime-tuple law is conjectural; the finite identity (C.2) supplies no estimate for its sieve remainder.

APPENDIX D. FIBONACCI, HURWITZ AND LERCH PROBES

Let $F_1 = F_2 = 1$, $F_{j+2} = F_{j+1} + F_j$ denote the Fibonacci numbers, distinguished by their subscript from the Dirichlet function $F_q(s)$. For $q \geq 2$ and integer $m \geq 0$, the reciprocal moment is

$$\mathcal{S}_{q,m} = \sum_p \frac{h_{F,q}(p)^m}{F_{h_{F,q}(p)}} = \sum_{j \geq 1} \frac{b_{q,j} j^m}{F_j}.$$

It converges absolutely by $b_{q,j} \leq f_q(j)$ and exponential growth of F_j . Put $\phi = (1 + \sqrt{5})/2$ and $r_\ell = (-1)^\ell \phi^{-(2\ell+1)}$. Binet's formula and a geometric expansion give

$$\frac{1}{F_j} = \sqrt{5} \sum_{\ell \geq 0} r_\ell^j,$$

since $F_j = \phi^j (1 - (-1)^j \phi^{-2j}) / \sqrt{5}$. Absolute convergence permits the interchange, yielding the original reciprocal-Fibonacci mechanism [2] with prime weights:

$$(D.1) \quad \mathcal{S}_{q,m} = \sqrt{5} \sum_{\ell \geq 0} [(z d/dz)^m G_{F,q}(z)]_{z=r_\ell}.$$

The interchange can be checked directly by majorizing $\sum_\ell |r_\ell|^j \leq \phi^{-j}/(1 - \phi^{-2})$ and then summing the polynomial fibre bound against that exponential.

Induction in the Fibonacci recurrence gives $F_j \geq \phi^{j-2}$. Thus a cutoff at M has the explicit tail bound

$$(D.2) \quad 0 \leq \mathcal{S}_{q,m} - \sum_{j \leq M} \frac{b_{q,j} j^m}{F_j} \leq q \phi^2 \sum_{j > M} (j+1)^{q-1} j^m \phi^{-j}.$$

The right side is a polynomial geometric tail and hence a finite combination of rational functions of ϕ^{-1} , obtained by repeated Euler differentiation of $z^{M+1}/(1-z)$. Ordinary finite Fibonacci kernels, such as $\sum_{p \leq N} F_{h_{F,q}(p)+u}$ with defined indices, follow directly from Proposition 2.2; their unreciprocal infinite positive versions diverge.

Similarly define the joint Hurwitz–Lerch probe

$$\mathcal{Q}_q(z, s, y) = \sum_{j \geq 1} b_{q,j} \frac{z^j}{(j+y)^s}, \quad y \geq 0.$$

For $|z| < 1$ it is entire in s ; for $z = 1$ its defining series converges on $\Re s > q$. These facts follow from the polynomial fibre bound. Euler differentiation in z gives prime-block moments, and differentiation in y gives shifted reciprocal moments, normally on their corresponding domains. The boundary result for its geometric specialization explains why the finite rational forms of unweighted polynomial counts do not persist after prime restriction.

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TEACHER OF MATHEMATICS, THE CORNERSTONE ACADEMY, POOLE, UNITED KINGDOM