

FLOOR AND CEILING IN RATIONALITY PROBLEMS AND CERTIFIED DENOMINATOR EXCLUSIONS

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ABSTRACT. Floor and ceiling functions detect exact integer hits and count rational candidates in certified intervals. A positive telescoping construction yields three absolutely convergent series with ordered partial sums and equal totals precisely at rational inputs. At a noninteger rational input, termwise order reverses once; the first two series converge nonuniformly. A complementary obstruction shows that globally defined finite rounded expressions cannot, under complete termwise ordering, have equal totals exactly on the rationals. Algebraic specifications permit ordered certificates, whereas arbitrary approximation programs do not admit a universal terminating rationality decision. For finite enclosures, determinant-one adjacent fractions certify the exact minimum denominator of a rational in the interval. Directed enclosures for $\pi + e$, $\zeta(7)$ and γ give reproducible denominator exclusions, including computations at a 1 000 000-place decimal scale. A small worked example connects enclosure construction, candidate generation and exact verification. The zeta evaluator uses a directed recurrence based on a classical harmonic-sum identity. Rational approximation explains an inverse-square-root denominator scale under explicit hypotheses, and counterexamples exclude a universal precision law. The finite exclusions are conditional bounds and do not determine rationality.

1. INTRODUCTION

The difference $\lceil t \rceil - \lfloor t \rfloor$ answers a simple arithmetic question: is t an integer? This observation suggests two approaches to rationality. An exact integer hit $qx \in \mathbb{Z}$ certifies that x is rational. If only an enclosure $L \leq x \leq U$ is available, integer counts in $[qL, qU]$ can instead exclude possible denominators. The same rounded arithmetic underlies both approaches, although their conclusions are different.

The main conceptual pair is Theorem 3.1 and Theorem 4.4. The first gives positive series with ordered partial sums and rationality detected by equality of their totals. At each noninteger rational input there is one compensating reversed summand. The second shows why complete termwise order is incompatible with that target within the stated class of globally defined finite rounded expressions. The construction and obstruction should therefore be read together: one exception to termwise order suffices for the explicit all-real construction, while retaining positivity and order of every partial sum.

The information supplied with the input changes the conclusion that is available. A defining integer polynomial supports a termwise ordered certificate by the rational-root denominator argument. An arbitrary approximation program does not support a universal finite rationality decision. A certified rational interval supports a different finite conclusion: exclusion of denominators below its exact interval minimum. These cases are separated throughout.

The certification component supplies a small complete example before presenting larger computations. The example derives an enclosure, generates a candidate, checks an adjacent-fraction certificate and states exactly what follows. The same certificate is applied to defining-series enclosures for $\pi + e$ and $\zeta(7)$ at scales 10^{200} , 10^{1000} and 10^{10000} , and to FLINT/Arb enclosures for these two inputs and γ at $D = 1\,000\,000$. The records

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give conditional lower bounds on a reduced rational denominator; they do not decide rationality. The half-precision discussion gives an estimate under explicit hypotheses and counterexamples to a universal rate.

The floor- and ceiling-indexed summation identities in [12, 13] provide the inverse-count framework used in Section 2.1. A level-counting specialization, with its boundary conventions, yields the integer-hit identity and Corollary 2.2. The telescoping construction is proved separately, and the finite denominator bounds use interval certificates.

Integer-function identities, continued fractions and the Stern–Brocot construction have their classical background in [5, 7]. Euler’s transformation is described in [10, Section 3.9(ii)]. The termwise-order obstruction uses density and Baire category [1, 9]. The harmonic-sum identity underlying the zeta recurrence is due to Dilcher and is recorded explicitly in [3, 6].

Throughout, $\mathbb{N} = \{1, 2, \dots\}$, and a rational representation p/q is reduced unless stated otherwise, with $q > 0$. All sums are ordinary convergent series. The notation (A_f, A_g, A_c) is reused only when a new triple is explicitly defined. A precision scale 10^D specifies a rounding grid; the actual enclosure width is recorded separately.

2. INTEGER HITS AND INVERSE COUNTS

Put

$$\delta(t) = \lceil t \rceil - \lfloor t \rfloor, \quad \tau(t) = 1 - \delta(t). \quad (1)$$

Thus $\delta(t)$ is zero at integers and one elsewhere, while $\tau(t)$ is the indicator of the integers. A related zero indicator is

$$Z(t) = 1 - \delta\left(\frac{t}{1 + |t|}\right) = 1 + \left\lfloor \frac{t}{1 + |t|} \right\rfloor - \left\lceil \frac{t}{1 + |t|} \right\rceil. \quad (2)$$

The fraction in this expression lies in $(-1, 1)$ and is an integer only at zero. Consequently $Z(0) = 1$ and $Z(t) = 0$ for $t \neq 0$.

2.1. Counting the levels of a power sequence. For $a, b > 0$, set $T_n = (bn)^a$. The following form of the inverse-count method isolates exactly the part of [12, 13] used here.

Proposition 2.1. *For $t \geq 0$, let*

$$N_{<}(t) = \max\left(0, \left\lceil \frac{t^{1/a}}{b} \right\rceil - 1\right), \quad N_{\leq}(t) = \left\lfloor \frac{t^{1/a}}{b} \right\rfloor.$$

The numbers of positive integers n satisfying $\lfloor T_n \rfloor = j$ and $\lceil T_n \rceil = j$ are, respectively,

$$m_f(j) = N_{<}(j+1) - N_{<}(j) \quad (j \geq 0), \quad (3)$$

$$m_c(j) = N_{\leq}(j) - N_{\leq}(j-1) \quad (j \geq 1). \quad (4)$$

For $0 < z < 1$, the absolutely convergent sums

$$H_f(z) = \sum_{n \geq 1} z^{\lfloor T_n \rfloor}, \quad H_c(z) = \sum_{n \geq 1} z^{\lceil T_n \rceil}$$

satisfy

$$H_c(z) - zH_f(z) = (1-z) \sum_{n \geq 1} \tau(T_n) z^{T_n} = (1-z) \sum_{j \geq 1} \tau(j^{1/a}/b) z^j. \quad (5)$$

Proof. Strict monotonicity of the power map gives $T_n < t$ if and only if $n < t^{1/a}/b$. Counting positive integers proves the formula for $N_{<}$, including $N_{<}(0) = 0$. Similarly, $T_n \leq t$ is equivalent to $n \leq t^{1/a}/b$. Subtracting successive counts gives (3) and (4). In particular, the zero floor level has multiplicity $\lceil 1/b \rceil - 1$; it must not be discarded when $b < 1$.

The multiplicities grow at most polynomially in j , so their generating series converge absolutely for $0 < z < 1$. They may therefore be regrouped as

$$H_f(z) = \sum_{j \geq 0} m_f(j)z^j, \quad H_c(z) = \sum_{j \geq 1} m_c(j)z^j.$$

If T_n is nonintegral, then $\lceil T_n \rceil = \lfloor T_n \rfloor + 1$, and its contribution to $H_c - zH_f$ vanishes. At an integer value $T_n = j$, the contribution is $(1 - z)z^j$. Finally, $T_n = j$ is equivalent to $n = j^{1/a}/b$, which is an admissible index exactly when it is a positive integer. This proves (5). $\square$

This proof is a level-counting specialization of the finite and infinite summation frameworks in [12, 13]. It uses absolute convergence to justify regrouping and makes no assertion about a limiting inverse-rounding error.

Corollary 2.2. *Let $x > 0$, $a = 1$ and $b = 1/x$. Then*

$$H_c(z) - zH_f(z) = \begin{cases} (1 - z)z^q/(1 - z^q), & x = p/q \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases} \quad (6)$$

Proof. In the last sum of (5), the condition is $jx \in \mathbb{Z}$. For $x = p/q$ reduced, this is equivalent to $q \mid j$. For an irrational x it never occurs. Summing the resulting geometric series proves the claim. $\square$

The direct mathematical use of Proposition 2.1 is the specialization in Corollary 2.2: choosing $a = 1$ and $b = 1/x$ turns the inverse-index integer condition into $jx \in \mathbb{Z}$. The general power-level counts (3)–(4) exhibit the connection with the summation framework of [12, 13]. The integer-hit identity (5) also has the direct termwise cancellation proof given above; its rationality content does not require the full level-counting machinery.

Equation (6) is an exact infinite characterization. Its zero value at an irrational input cannot be certified merely by checking finitely many missing hits. Section 3 uses the same integer-hit criterion in a different telescoping construction, proved independently, to give the equality-versus-strict-inequality orientation of three sums.

The finite method of Section 6 instead counts candidate numerators inside rational intervals. The enclosure methods of Section 7 and the denominator exclusions of Section 8 use those interval certificates. Their proofs and computations do not depend on the power-level counts of Proposition 2.1.

3. THREE SUMS WITH ORDERED PARTIAL SUMS

For a real x , define

$$P_0(x) = 1, \quad P_n(x) = \prod_{k=1}^n \delta(kx), \quad d_n(x) = P_{n-1}(x) - P_n(x). \quad (7)$$

If x is irrational, every $P_n(x)$ is one. If $x = p/q$ is rational, $P_n(x) = 1$ for $n < q$ and $P_n(x) = 0$ for $n \geq q$. In the rational case, $d_q = 1$ is the only nonzero d_n .

Set $G_0 = 0$ and, for $n \geq 1$, $G_n = 1 + P_n/2$. The separate definition of G_0 is essential. Define

$$\begin{aligned} A_f(n, x) &= 2^{-n} + d_n + \lfloor G_n \rfloor - \lfloor G_{n-1} \rfloor, \\ A_g(n, x) &= 2^{-n} + d_n + G_n - G_{n-1}, \\ A_c(n, x) &= 2^{-n} + d_n + \lceil G_n \rceil - \lceil G_{n-1} \rceil. \end{aligned} \quad (8)$$

Theorem 3.1. *For every real x , all terms in (8) are positive and all three series converge absolutely. Their partial sums satisfy*

$$\begin{aligned} S_f(N, x) &= 3 - 2^{-N} - P_N(x), \\ S_g(N, x) &= 3 - 2^{-N} - P_N(x)/2, \\ S_c(N, x) &= 3 - 2^{-N}. \end{aligned} \tag{9}$$

In particular, $S_f(N, x) \leq S_g(N, x) \leq S_c(N, x)$ for every N , and

$$(S_f, S_g, S_c) = \begin{cases} (3, 3, 3), & x \in \mathbb{Q}, \\ (2, 5/2, 3), & x \notin \mathbb{Q}. \end{cases} \tag{10}$$

For a noninteger rational p/q , termwise order reverses exactly once, at $n = q$.

Proof. For $n \geq 1$, $G_n \in \{1, 3/2\}$, so $\lfloor G_n \rfloor = 1$ and $\lceil G_n \rceil = 1 + P_n$. The geometric baseline sums to $1 - 2^{-N}$, the d_n sum to $1 - P_N$, and the remaining differences telescope. This gives (9), its ordering and the limits (10).

The first triple is $(5/2 - P_1, 5/2 - P_1/2, 5/2)$. For $n \geq 2$ the terms are

$$(A_f, A_g, A_c) = (2^{-n} + d_n, 2^{-n} + d_n/2, 2^{-n}). \tag{11}$$

They are positive and have a summable geometric baseline with at most one additional contribution. If $q > 1$, the exceptional triple is

$$(2^{-q} + 1, 2^{-q} + 1/2, 2^{-q}),$$

which has the reversed strict order. Every other triple with $n \geq 2$ is equal. At an integer input, the first triple is equal and there is no later exception. $\square$

For example, $x = 2/3$ has first triple $(3/2, 2, 5/2)$, second triple $(1/4, 1/4, 1/4)$ and third triple $(9/8, 5/8, 1/8)$. All subsequent triples are equal, and the three totals are three. An irrational input has the same first two triples, but it never has a compensating integer hit.

3.1. The convergence qualification.

Proposition 3.2. *The floor and middle series in (8) do not converge uniformly as functions of $x \in \mathbb{R}$. For any N and any rational input of denominator $q > N$,*

$$S_f - S_f(N) = 1 + 2^{-N}, \quad S_g - S_g(N) = 1/2 + 2^{-N}.$$

At an irrational input these tails are both 2^{-N} ; the ceiling tail is always 2^{-N} .

Proof. Subtract (9) from (10). Rationals with $q > N$ exist for every N , so the suprema of the first two tails do not tend to zero. $\square$

Before a hit, the values $P_1, \dots, P_N$ are identical for every irrational input and for every reduced rational whose denominator exceeds N . A reversal certifies a noninteger rational input, but its absence over a finite prefix certifies neither outcome. Positivity of all summands does not remove this distinction between pointwise convergence and a uniform stopping rule.

4. TERMWISE BOUNDS AND THE INFORMATION SUPPLIED WITH THE INPUT

Lemma 4.1. *Suppose three series have finite sums and satisfy $A_f(n, x) \leq A_g(n, x) \leq A_c(n, x)$ for every n . Their totals coincide if and only if all three terms coincide at every index.*

Proof. The differences $A_g - A_f$ and $A_c - A_g$ are nonnegative. If the sums of these differences vanish, every difference vanishes. The converse is immediate. $\square$

For a fixed real argument t and a positive mesh Δ , the bounds

$$\Delta \lfloor t/\Delta \rfloor \leq t \leq \Delta \lceil t/\Delta \rceil$$

each have error less than Δ . Both therefore approach t as $\Delta \rightarrow 0$, for rational and irrational t alike.

The equality condition for a weighted envelope also needs its exact form stated. Let $w_n > 0$ and let ϕ be strictly increasing. Consider

$$w_n \phi(\lfloor T_n(x) \rfloor), \quad w_n \phi(T_n(x)), \quad w_n \phi(\lceil T_n(x) \rceil).$$

Assuming the three series converge, their totals are equal if and only if every $T_n(x)$ is an integer. Indeed, the terms are ordered; Lemma 4.1 makes the terms equal, and injectivity of ϕ forces $T_n(x) = \lfloor T_n(x) \rfloor$. The converse is immediate. For example, $\phi(t) = 1/2 + \pi^{-1} \arctan t$ takes values in $(0, 1)$, so summable positive weights guarantee positive convergent envelopes. This conclusion is for the displayed envelope form, not for arbitrary floor–ceiling constructions. Section 3 allows its single compensating reversal.

4.1. A specified polynomial. An algebraic specification supplies information that decimal approximations alone do not provide. For a root of an integer polynomial there is a finite certificate, expressible in floor–ceiling form.

Proposition 4.2. *Let x be a real root of*

$$a_m t^m + \cdots + a_0 \in \mathbb{Z}[t], \quad a_m \neq 0,$$

and put $B = |a_m|$. Then $x \in \mathbb{Q}$ if and only if $Bx \in \mathbb{Z}$. With

$$u = 1 + Bx - \lfloor Bx \rfloor, \quad (A_f, A_g, A_c) = 2^{-n}(\lfloor u \rfloor, u, \lceil u \rceil),$$

the terms are positive and ordered, and their totals are

$$\begin{cases} (1, 1, 1), & x \in \mathbb{Q}, \\ (1, 1 + \{Bx\}, 2), & x \notin \mathbb{Q}, \end{cases}$$

where $\{t\} = t - \lfloor t \rfloor$.

Proof. If $x = p/q$ is reduced, multiplying the polynomial equation by q^m shows that q divides $a_m p^m$. Since p and q are coprime, $q \mid a_m$, so Bx is an integer. The converse follows from $x = (Bx)/B$. Therefore $u = 1$ at a rational root and $1 < u < 2$ at an irrational root. The asserted totals follow from $\sum_{n \geq 1} 2^{-n} = 1$. $\square$

For $x = \sqrt{2}$, the polynomial is $t^2 - 2$, with $B = 1$. Its positive root lies strictly between one and two and is not an integer. Proposition 4.2 therefore proves its irrationality and gives the totals $(1, \sqrt{2}, 2)$. This calculation exploits an exact defining equation, rather than a search for a missing decimal hit.

4.2. A construction on all algebraic inputs. For completeness, polynomial certificates can also be enumerated. List the nonconstant polynomials in $\mathbb{Z}[t]$ as $p_1, p_2, \dots$, and let B_j be the absolute value of the leading coefficient of p_j . Define

$$E_j(x) = Z(p_j(x))\delta(B_j x), \quad T_0 = 1, \quad T_j = \prod_{i=1}^j (1 - E_i(x)), \quad u_j = 1 + (1 - T_j)/2.$$

All these are finite expressions at each index. The ordered triple

$$(A_f(j, x), A_g(j, x), A_c(j, x)) = 2^{-j}(\lfloor u_j \rfloor, u_j, \lceil u_j \rceil) \tag{12}$$

has equal totals one at every rational input. At an irrational algebraic input, let m be the first index with $E_m = 1$. Its totals are

$$(1, 1 + 2^{-m}, 1 + 2^{1-m}). \tag{13}$$

Indeed, Proposition 4.2 makes every gate zero at a rational input; an irrational algebraic root opens at least one gate. Before m , $u_j = 1$; from m onward, $u_j = 3/2$, and summing the geometric tails proves (13). At a transcendental input every gate is zero, so the totals are again $(1, 1, 1)$. Thus the algebraic domain restriction is indispensable for this construction.

With a defining polynomial and a rational isolating interval, exact root comparisons and polynomial equality tests are effective. The same zero tests need not be decidable when an arbitrary real number is supplied only by successive rational enclosures. The representation of the input is part of any algorithmic claim [15].

4.3. An obstruction for finite expressions on all real inputs. Here a finite rounded expression means an expression obtained from continuous real functions and constants by finitely many continuous arithmetic operations, floors and ceilings. Every intermediate value must be finite and defined wherever the expression is used; in particular, denominators in divisions must be nonzero. This class excludes an infinite limit hidden inside a single summand.

Lemma 4.3. *A finite rounded expression defined on all of $\mathbb{R}$ is continuous on some dense open subset of $\mathbb{R}$.*

Proof. The assertion follows by induction on the expression. Continuous operations preserve continuity on the intersection of finitely many dense open sets. Suppose an intermediate expression g is continuous on a dense open set V . Within V , discard the boundaries of the level sets $\{g = k\}$ for $k \in \mathbb{Z}$. Each such boundary is closed in V and has empty interior there. The family is locally finite: near a point of V , continuity bounds g in a bounded interval, which meets only finitely many integer levels. Its union is therefore closed and has empty interior in V .

On the remaining open dense subset, either g avoids the integers in a neighborhood, or it is identically an integer in a neighborhood. Both $\lfloor g \rfloor$ and $\lceil g \rceil$ are locally constant there. This proves the induction step. An open subset dense in the dense open set V is also open and dense in $\mathbb{R}$. $\square$

A set is *nowhere dense* when its closure has empty interior; it is *meagre* when it is a countable union of nowhere dense sets. Its complement is then called *comeagre*.

Theorem 4.4. *Suppose each term of three convergent series is a finite rounded expression defined on all real inputs, and $A_f \leq A_g \leq A_c$ termwise. If their totals coincide at every rational input, then they also coincide on a dense, comeagre set of irrational inputs. In particular, this class cannot give termwise ordered series with equal totals exactly on $\mathbb{Q}$.*

Proof. By Lemma 4.1, the two termwise gaps vanish at every rational input. For each index n , Lemma 4.3 gives a dense open set V_n on which all three terms are continuous. Density of the rationals implies that the gaps vanish throughout V_n .

Let $G = (\bigcap_{n \geq 1} V_n) \setminus \mathbb{Q}$. Its complement is the union of the closed nowhere dense sets $\mathbb{R} \setminus V_n$ and the countable set $\mathbb{Q}$, so G is comeagre. It is also dense. To see this directly, enumerate the rationals as $r_1, r_2, \dots$ and begin with any nondegenerate closed interval J_0 inside any specified open interval. Successively choose nondegenerate closed intervals J_n of length at most 2^{-n} , with J_n contained in the interior of J_{n-1} and in $V_n \setminus \{r_n\}$. Density and openness allow every choice. Completeness gives a point in all the J_n , hence in G and the specified open interval. At every point of G , all three terms, and therefore their totals, coincide. $\square$

The proof uses the classical density and Baire-category mechanism [1, 9]. A closely related obstruction already follows for Baire-class-one gaps. Baire-class-one functions are pointwise limits of continuous functions. For real-valued functions on $\mathbb{R}$, this class has the standard property that inverse images of open sets are countable unions of closed

sets [9, Section 24]. If such a gap vanishes on $\mathbb{Q}$, its nonzero set is a countable union of closed sets avoiding $\mathbb{Q}$. Each has empty interior by density, and is therefore nowhere dense. The nonzero set is meagre; a countable family of such gaps vanishes simultaneously on a comeagre set. Removing $\mathbb{Q}$ still leaves a dense comeagre set of irrational points.

This comparison identifies the classical topological principle behind the obstruction. Lemma 4.3 separately verifies the stronger dense-open continuity property for the finite-expression syntax used here; it does not assume that arbitrary compositions of Baire-class-one functions remain in that class. The lemma supplies the continuity property needed for the stated class of expressions.

The theorem concerns its stated expression class. The polynomial construction has a restricted input domain, and the universal construction has a reversed term. Neither contradicts the obstruction.

5. WHAT A FINITE COMPUTATION CAN CERTIFY

A finite rational interval of positive width contains both rational and irrational numbers. For the rational case, choose a grid spacing smaller than its width; an integer multiple of that spacing lies inside. For the irrational case, add a sufficiently small rational multiple of $\sqrt{2}$ to an interior rational point. Thus an enclosure alone does not classify rationality without additional structural information.

There is also a general obstruction when the input is a program that computes arbitrarily accurate rational approximations. It does not prevent successful algorithms on specified families such as polynomial roots.

Proposition 5.1. *There is no algorithm that, for every program computing a real number with a certified error at most 2^{-n} on request n , always halts and correctly decides whether that number is rational.*

Proof. Given a machine M , define

$$x_M = \begin{cases} 2^{-T}\sqrt{2}, & M \text{ first halts at step } T \geq 1, \\ 0, & M \text{ never halts.} \end{cases}$$

A program for approximating x_M can be constructed uniformly from M . On request n , simulate M for $n + 2$ steps. If it has halted, approximate the indicated algebraic value to error 2^{-n} . If it has not, output zero. Any later halt has $T > n + 2$, so the error of that output is less than 2^{-n} , whether or not a later halt occurs. Square-root approximations themselves can be computed by rational bisection.

The value x_M is rational exactly when M never halts. A universal rationality decider applied to these approximation programs would therefore decide the halting problem, contradicting its undecidability [14]. This is a statement about program inputs, not about a symbolic representation permitting exact comparisons. $\square$

The constructive alternative is to ask a finite question that an interval really can answer: what is the smallest denominator of any rational number it contains?

6. CERTIFIED INTERVALS AND MINIMUM DENOMINATORS

Let $I = [L, U]$, where $L < U$ are rational. For $q \in \mathbb{N}$, define

$$H_I(q) = \lfloor qU \rfloor - \lfloor qL \rfloor + 1. \quad (14)$$

It counts the integers in $[qL, qU]$ and is nonnegative. Put

$$q_{\min}(I) = \min\{q \in \mathbb{N} : H_I(q) > 0\}. \quad (15)$$

The minimum exists: for $q \geq \lceil 1/(U - L) \rceil$, the interval $[qL, qU]$ has length at least one and contains an integer. It is also the least reduced denominator of any rational in I . A

fraction with the first admissible denominator must already be reduced, since reduction otherwise gives an earlier admissible denominator.

If a real $x \in I$ has a reduced rational representation p/q , then

$$q \geq q_{\min}(I). \quad (16)$$

Computing this threshold is an exclusion of finitely many denominators. It supplies no upper bound on a hypothetical denominator of x .

6.1. A short exact certificate.

Proposition 6.1. *Suppose $b, d > 0$ and integers a, b, c, d satisfy*

$$bc - ad = 1, \quad \frac{a}{b} < L \leq \frac{a+c}{b+d} \leq U < \frac{c}{d}. \quad (17)$$

Then $(a+c)/(b+d)$ is reduced and

$$q_{\min}([L, U]) = b + d.$$

Proof. Any fraction p/q strictly between a/b and c/d has positive integer values $cq - dp$ and $bp - aq$. The determinant identity gives

$$q = b(cq - dp) + d(bp - aq) \geq b + d. \quad (18)$$

The mediant in (17) attains this denominator. A common divisor of $a+c$ and $b+d$ would divide $b(a+c) - a(b+d) = bc - ad = 1$, so the mediant is reduced. $\square$

Example 6.2. For $I = [31/100, 32/100]$, use $a/b = 4/13$ and $c/d = 1/3$. Their determinant is one, they strictly bracket I , and their mediant is $5/16 \in I$. Thus $q_{\min}(I) = 16$, certified without checking each smaller denominator. In particular, $H_I(15) = 0$ and $H_I(16) = 1$.

6.2. The compressed Stern–Brocot search. The mediant construction is classical [5]. We give an exact interval version and derive its jump formulas to handle closed endpoints correctly.

First, if $\lceil L \rceil \leq U$, return the integer $\lceil L \rceil$ and the minimum denominator one. Otherwise set $h = \lfloor L \rfloor$ and write

$$L - h = \lambda/\rho, \quad U - h = \mu/\sigma, \quad 0 < \lambda/\rho < \mu/\sigma < 1,$$

with positive denominators. Start with $(a, b, c, d) = (0, 1, 1, 0)$, where $1/0$ is a right sentinel. At each step form $(a+c)/(b+d)$.

If this mediant is below λ/ρ , make the left update

$$k = \left\lfloor \frac{\lambda b - a\rho - 1}{c\rho - \lambda d} \right\rfloor, \quad (a, b) \leftarrow (a + kc, b + kd). \quad (19)$$

If it is above μ/σ , make the right update

$$k = \left\lfloor \frac{c\sigma - \mu d - 1}{\mu b - a\sigma} \right\rfloor, \quad (c, d) \leftarrow (c + ka, d + kb). \quad (20)$$

If the mediant belongs to the closed target interval, return

$$\frac{a + c + h(b + d)}{b + d}. \quad (21)$$

Proposition 6.3. *The denominators in the jump ratios are positive, each applied jump has $k \geq 1$, and the algorithm terminates. Its return value has the least possible denominator in $[L, U]$ and supplies the certificate of Proposition 6.1 whenever the integer case did not apply.*

Proof. The invariant is $bc - ad = 1$, with the left fraction strictly below the shifted interval and the right fraction strictly above it. A left update of size k remains strictly below its endpoint precisely when

$$k(cp - \lambda d) < \lambda b - a\rho.$$

All quantities are integers and the coefficient of k is positive. The greatest permissible integer k is therefore (19). A mediant below the endpoint makes $k = 1$ permissible. The same calculation on the right gives (20). The subtraction of one in each numerator preserves strict bracketing even when the unadjusted quotient is integral. Both updates preserve the determinant.

For termination, fix any reduced rational m/r in the shifted target interval. While it is bracketed, its coordinates

$$\alpha = cr - dm > 0, \quad \beta = bm - ar > 0$$

are positive integers, with $r = b\alpha + d\beta$. A left jump replaces β by $\beta - k\alpha$; a right jump replaces α by $\alpha - k\beta$. Bracketing keeps the new coordinate positive, while $\alpha + \beta$ decreases. Thus only finitely many jumps are possible before a mediant enters the interval. Since the shifted interval is inside $(0, 1)$, the right sentinel has already acquired a positive denominator by that time. Proposition 6.1, followed by the integer shift, proves minimality. $\square$

6.3. Nested certificates and an infinite characterization.

Proposition 6.4. *Let I_n be nested bounded closed rational intervals of positive width, with widths tending to zero and intersection $\{x\}$. Then $q_{\min}(I_n)$ is nondecreasing. If $x = p/q$ is reduced rational, it eventually equals q . If x is irrational, it tends to infinity.*

Proof. Nesting cannot introduce a smaller-denominator rational. For any fixed M , there are only finitely many reduced rationals of denominator at most M in I_1 . Every one different from x is eventually absent from I_n . If $x = p/q$, apply this observation to $M = q$ and use the presence of x at every index. If x is irrational, apply it separately for every M . $\square$

This gives an infinite rationality characterization through finite certificates. Finite growth does not certify unbounded future growth, and a finite plateau does not certify eventual stabilization. It is the interval counterpart of the stopping issue in Proposition 3.2.

7. EXACT ENCLOSURES AND OUTWARD ROUNDING

We now construct intervals using rational arithmetic only. Floors and ceilings enter as directed rounding operations; each analytic remainder is bounded before any denominator search is undertaken.

7.1. An enclosure for $\pi + e$. Let $B = 10^D$. For $r \in \{5, 239\}$, let K_r be the least nonnegative integer satisfying

$$(2K_r + 1)r^{2K_r+1} > 1000B, \quad t_{r,k} = \frac{(-1)^k}{(2k+1)r^{2k+1}}.$$

Define scaled integers

$$l_r = \sum_{k=0}^{K_r-1} \lfloor Bt_{r,k} \rfloor - \mathbf{1}_{\{K_r \text{ odd}\}}, \quad (22)$$

$$u_r = \sum_{k=0}^{K_r-1} \lceil Bt_{r,k} \rceil + \mathbf{1}_{\{K_r \text{ even}\}}. \quad (23)$$

Here $\mathbf{1}$ denotes the indicator of the stated condition. The arctangent series has a remainder of sign $(-1)^{K_r}$ and magnitude less than $1/(1000B)$. Therefore

$$l_r/B < \arctan(1/r) < u_r/B.$$

One way to justify both properties directly is to integrate the finite geometric expansion of $1/(1+t^2)$ from zero to $1/r$; its remainder has the claimed sign and is bounded by the integral of t^{2K_r} .

Let M be the least positive integer with $M! > 1000B$ and put

$$l_e = \sum_{k=0}^{M-1} \left\lfloor \frac{B}{k!} \right\rfloor, \quad u_e = \sum_{k=0}^{M-1} \left\lceil \frac{B}{k!} \right\rceil + \left\lceil \frac{2B}{M!} \right\rceil. \quad (24)$$

Every contribution to u_e , including the tail allowance $\lceil 2B/M! \rceil$, is rounded upward. The positive exponential tail satisfies

$$0 < e - \sum_{k=0}^{M-1} \frac{1}{k!} < \frac{1}{M!} \sum_{j \geq 0} (M+1)^{-j} \leq \frac{2}{M!},$$

so $l_e/B < e < u_e/B$.

Machin's identity gives

$$\pi = 16 \arctan(1/5) - 4 \arctan(1/239).$$

For a brief verification, if $u = \arctan(1/5)$ and $v = \arctan(1/239)$, the double-angle formulas give $\tan(2u) = 5/12$ and $\tan(4u) = 120/119$. Hence $\tan(4u - v) = 1$. The inequality $\tan(2u) < 1$ gives $2u < \pi/4$, while $v < u$ gives $4u - v > 0$. Thus $0 < 4u - v < \pi/2$, and it equals $\pi/4$.

Thus the explicit interval

$$L_D = \frac{16l_5 - 4u_{239} + l_e}{B}, \quad U_D = \frac{16u_5 - 4l_{239} + u_e}{B} \quad (25)$$

satisfies $L_D < \pi + e < U_D$. Individual rounding accumulates a width larger than one grid unit; the computed scaled width $B(U_D - L_D)$ is included in Table 1.

7.2. A positive transformed zeta series. For an integer $s \geq 2$, write $\zeta(s) = \sum_{n \geq 1} n^{-s}$. The following derivation specializes Euler's transformation [10] and supplies the positivity and tail estimates needed for directed rounding. The integer parameter also explains why only six coefficient updates are needed when $s = 7$.

Proposition 7.1. *For integers $s \geq 2$ and $k \geq 0$, put*

$$d_{k,s} = \sum_{j=0}^k \frac{(-1)^j \binom{k}{j}}{(j+1)^s}. \quad (26)$$

Then $0 < d_{k,s} \leq 1$, and, with $\kappa_s = 1 - 2^{1-s}$,

$$\zeta(s) = \kappa_s^{-1} \sum_{k \geq 0} \frac{d_{k,s}}{2^{k+1}}. \quad (27)$$

For $K \geq 1$, its first K terms have remainder

$$0 < \zeta(s) - \kappa_s^{-1} \sum_{k=0}^{K-1} \frac{d_{k,s}}{2^{k+1}} < \kappa_s^{-1} 2^{-K}. \quad (28)$$

Proof. Substituting $t = \exp(-u)$ and integrating by parts gives

$$\frac{1}{(j+1)^s} = \frac{1}{(s-1)!} \int_0^1 t^j (-\log t)^{s-1} dt.$$

The binomial theorem therefore yields

$$d_{k,s} = \frac{1}{(s-1)!} \int_0^1 (1-t)^k (-\log t)^{s-1} dt. \quad (29)$$

It is positive, equals one for $k = 0$, and is less than one for $k \geq 1$. By monotone convergence,

$$\sum_{k \geq 0} \frac{d_{k,s}}{2^{k+1}} = \frac{1}{(s-1)!} \int_0^1 \frac{(-\log t)^{s-1}}{1+t} dt.$$

Expanding $(1+t)^{-1}$ gives $\sum_{j \geq 0} (-1)^j / (j+1)^s$. Interchanging the integrals and that series is justified by the summability of $\sum_{j \geq 0} (j+1)^{-s}$. Separating even from odd indices identifies this alternating sum as $\kappa_s \zeta(s)$. Finally, the remaining positive terms are strictly bounded by $\kappa_s^{-1} \sum_{k \geq K} 2^{-k-1} = \kappa_s^{-1} 2^{-K}$. $\square$

7.3. A coefficient recurrence and its rounded version. Let $h_r(n)$ be the coefficient of z^r in

$$\prod_{j=1}^n (1 - z/j)^{-1}, \quad h_0(n) = 1, \quad h_r(0) = 0 \quad (r \geq 1). \quad (30)$$

Multiplication by the final factor, or equivalently multiplication by $1 - z/n$ on the other side, gives

$$h_r(n) = h_r(n-1) + h_{r-1}(n)/n. \quad (31)$$

The coefficients at the new index n must therefore be updated in increasing order of r .

Expanding the product in (30) gives a complete homogeneous reciprocal sum. Its binomial form is Dilcher's established harmonic-sum identity [3]; the following explicit form is recorded in [6, eq. (5.2)]:

$$h_r(n) = \sum_{1 \leq j_1 \leq \dots \leq j_r \leq n} \frac{1}{j_1 \cdots j_r} = \sum_{j=1}^n \frac{(-1)^{j-1} \binom{n}{j}}{j^r} \quad (r \geq 1).$$

Since $\binom{n-1}{j-1} = (j/n) \binom{n}{j}$, the definition (26) gives

$$d_{n-1,s} = \frac{1}{n} \sum_{j=1}^n \frac{(-1)^{j-1} \binom{n}{j}}{j^{s-1}} = \frac{h_{s-1}(n)}{n}.$$

The beta-integral calculation below gives another derivation of the coefficient identity in the form needed for directed rounding.

For $\operatorname{Re} z < 1$, expanding $(1-t)^{n-1}$ and integrating gives

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{(-1)^j \binom{n-1}{j}}{j+1-z} &= \int_0^1 t^{-z} (1-t)^{n-1} dt \\ &= \frac{1}{n} \prod_{j=1}^n (1 - z/j)^{-1}. \end{aligned}$$

The second equality follows by repeated integration by parts. Comparing coefficients of z^{s-1} near zero proves

$$d_{n-1,s} = h_{s-1}(n)/n. \quad (32)$$

For $s = 7$, write $d_k = d_{k,7}$ and use $h_1, \dots, h_6$.

Choose $C = 10^{D+12}$ and the least $K \geq 1$ with $63 \cdot 2^K > 64C$. Initialize integer bounds

$$l_0 = u_0 = C, \quad l_r = u_r = 0 \quad (1 \leq r \leq 6).$$

For $n = 1, \dots, K$, update in increasing order $r = 1, \dots, 6$ by

$$l_r \leftarrow l_r + \lfloor l_{r-1}/n \rfloor, \quad u_r \leftarrow u_r + \lceil u_{r-1}/n \rceil. \quad (33)$$

Then set $v_n = \lfloor l_6/n \rfloor$ and $V_n = \lceil u_6/n \rceil$, and update integers initially zero by

$$S_l \leftarrow 2S_l + v_n, \quad S_u \leftarrow 2S_u + V_n. \quad (34)$$

Proposition 7.2. *After the K updates,*

$$\frac{64S_l}{63C \cdot 2^K} < \zeta(7) < \frac{64(S_u + C)}{63C \cdot 2^K}. \quad (35)$$

In particular, for $B = 10^D$ the integers

$$l = \left\lfloor \frac{64BS_l}{63C \cdot 2^K} \right\rfloor, \quad u = \left\lceil \frac{64B(S_u + C)}{63C \cdot 2^K} \right\rceil \quad (36)$$

give a certified interval $l/B < \zeta(7) < u/B$. The lower endpoint uses a floor and the upper endpoint a ceiling; thus the final grid width includes outward rounding.

Proof. Induction using (31) and the increasing order of updates gives $l_r \leq Ch_r(n) \leq u_r$ after every update. Hence $v_n \leq Cd_{n-1} \leq V_n$. The binary accumulator satisfies

$$\frac{S_l}{C \cdot 2^K} \leq \sum_{n=1}^K \frac{d_{n-1}}{2^n} \leq \frac{S_u}{C \cdot 2^K}.$$

The accumulator bounds are scaled by one factor of C . Multiply by $64/63 = \kappa_7^{-1}$ and use the strictly positive tail bound (28); adding C to the upper accumulator contributes exactly $64/(63 \cdot 2^K)$. Outward rounding gives (36). $\square$

No presumed rounding accuracy is used in this proof. The final width $u - l$ is computed exactly; it is one for all three scales reported below. This is the width of the final outward-rounded grid interval, not the unrounded tail allowance, and it is not asserted for every scale.

For fixed $s = 7$, the evaluator maintains six coefficient bounds and updates each coefficient at every transformed term. Integer sizes also grow with precision, so this statement does not give a bit-complexity bound. More rapidly convergent alternating-series accelerations are available, including those of Cohen, Rodriguez Villegas and Zagier [2]. The Euler representation is used here for its positive terms, explicit tail bound and transparent directed rounding; no claim of optimal speed is made.

7.4. Rigorous balls and exact decimal-grid rounding. For the million-place extension, the enclosure step uses Arb's rigorous ball arithmetic, now part of FLINT [4, 8]. The Python binding [11] represents a finite real ball as $[m - r, m + r]$, where the midpoint m and nonnegative radius r are exactly representable binary numbers. Evaluation and rounding errors are included in the radius. This provides an alternative enclosure method; the individually rounded series in the preceding subsections are not used at this scale.

For the Euler–Mascheroni constant, the defining limit is

$$\gamma = \lim_{N \rightarrow \infty} (H_N - \log N), \quad H_N = \sum_{j=1}^N \frac{1}{j}.$$

The evaluations are `arb.pi()+arb.const_e()` for $\pi + e$, `arb(7).zeta()` for $\zeta(7)$, and `arb.const_euler()` for γ . At $D = 1\,000\,000$, the first run requests $D + 64$ decimal digits, giving a working precision of 3 322 144 bits. A rerun requests $D + 96$ digits, giving 3 322 250 bits. The records retain the primary midpoint and radius as exact dyadic mantissas and exponents. The gamma record also serializes the higher-precision ball, allowing its containment in the primary ball and agreement of its grid endpoints to be checked by exact integer arithmetic. The following elementary lemma converts such a ball into a rational enclosure using only integer arithmetic.

Proposition 7.3. *Suppose $x \in [m - r, m + r]$, with $m = M2^E$, $r = R2^F$, $M \in \mathbb{Z}$, $R \in \mathbb{Z}_{\geq 0}$ and $E, F \in \mathbb{Z}$. For $B = 10^D$, put*

$$h = -\min(E, F, 0), \quad T = 2^h, \quad A_{\pm} = M2^{E+h} \pm R2^{F+h}. \quad (37)$$

Then A_-, A_+ are integers, and

$$l = \left\lfloor \frac{BA_-}{T} \right\rfloor, \quad u = \left\lceil \frac{BA_+}{T} \right\rceil \quad (38)$$

give $l/B \leq x \leq u/B$. If

$$lT < BA_-, \quad BA_+ < uT, \quad (39)$$

then the entire ball lies strictly inside $[l/B, u/B]$.

Proof. Both exponents $E + h, F + h$ are nonnegative, and $m \pm r = A_{\pm}/T$. The defining inequalities for floor and ceiling give

$$\frac{l}{B} \leq \frac{A_-}{T} \leq x \leq \frac{A_+}{T} \leq \frac{u}{B}.$$

The strict comparisons in (39) make the first and last inequalities strict. $\square$

For all three saved million-place records, exact integer arithmetic verifies (38), (39) and $u - l = 1$. The generation code rounds a scaled Arb ball outward; these checks verify that its saved endpoints also equal the exact floor and ceiling in (38). Thus the physical enclosure width is exactly $10^{-1000000}$. The connection between the evaluated ball and its defining constant uses Arb's enclosure guarantee; the conversion of that saved ball to the decimal grid is certified by the displayed integer identities.

8. REPRODUCIBLE DENOMINATOR EXCLUSIONS

8.1. A small end-to-end computation. Take the three-place grid $B = 1000$. The directed evaluator of Proposition 7.2, with $C = 10^{15}$ and the least cutoff $K = 50$, gives the exact scaled endpoints $l = 1008$, $u = 1009$. Thus it supplies the enclosure

$$\zeta(7) \in I = [1008/1000, 1009/1000].$$

The outward endpoints and cutoff are reproduced by `code/worked_example.py`; the analytic enclosure justification is the proposition, not a comparison with approximate decimal digits.

For candidate generation, subtract the integer part one. The shifted interval is $[8/1000, 9/1000]$. Starting from the sentinels $0/1$ and $1/0$, the first mediant lies above this interval. The right-jump formula (20) gives

$$k = \left\lfloor \frac{1000 - 1}{9} \right\rfloor = 111.$$

The new right endpoint is $1/111$, and the next mediant $1/112$ belongs to the shifted interval. Adding the integer part returns $p/q = 113/112$. Its unshifted parents are $a/b = 1/1$ and $c/d = 112/111$.

The complete minimum-denominator check is now short:

$$\begin{aligned} bc - ad &= 112 - 111 = 1, & aB &= 1000 < 1008 = lb, \\ ud &= 1009 \cdot 111 = 111999 < 112000 = cB, \\ lq &= 112896 \leq 113000 = pB \leq 113008 = uq, \\ p &= a + c = 113, & q &= b + d = 112. \end{aligned}$$

Proposition 6.1 therefore proves $q_{\min}(I) = 112$ without relying on the search's claim of minimality. As a small additional check, $H_I(111) = 0$ and $H_I(112) = 1$; the example program also exhaustively checks all $q < 112$.

The conclusion is conditional and finite: if $\zeta(7)$ has a reduced rational representation, its denominator is at least 112. The candidate 113/112 attains the interval minimum and is not identified with $\zeta(7)$. This is the same logical conclusion as in the larger computations below.

8.2. Defining-series records. The following records are larger illustrations of the process in Section 8.1. Apply the exact search of Section 6 to the intervals in (25) and (36). For each row, the certificate gives the exact minimum $q_{\min} = b + d$. Table 1 displays its leading six significant digits rounded down, rather than the complete integer. Each displayed threshold is strictly below that exact minimum, as verified by an integer comparison.

Input	D	C_D	Rounded-down threshold	Digits of $q_{\min}$
$\pi + e$	200	2533	2.33427×10^{98}	99
$\zeta(7)$	200	1	9.92094×10^{99}	100
$\pi + e$	1000	12670	9.61140×10^{497}	498
$\zeta(7)$	1000	1	1.52229×10^{500}	501
$\pi + e$	10000	126028	1.97688×10^{4997}	4998
$\zeta(7)$	10000	1	9.28497×10^{4999}	5000

TABLE 1. Exact enclosures have width $C_D 10^{-D}$. Each threshold is a six-significant-digit downward truncation of the certified integer $q_{\min} = b + d$. The last column gives the length of that exact integer. These are interval minima; their rational candidates need not equal the input.

Theorem 8.1. *Every reduced rational representation of $\pi + e$, if one exists, has denominator*

$$q > 1.97688 \times 10^{4997}.$$

Every reduced rational representation of $\zeta(7)$, if one exists, has denominator

$$q > 9.28497 \times 10^{4999}.$$

At the 1000-place scale the latter exclusion is $q > 1.52229 \times 10^{500}$. The other finite exclusions in Table 1 hold as well.

Proof. The defining-series bounds (25) and (36) place the inputs strictly between the rational endpoints. The supplementary integer records supply, for each row, the scaled endpoint integers l, u , the scale $B = 10^D$, the adjacent pairs $(a, b), (c, d)$, and the candidate (p, q) . Their checks are precisely

$$\begin{aligned} b, d > 0, \quad bc - ad = 1, \quad aB < lb, \quad ud < cB, \\ p = a + c, \quad q = b + d, \quad lq \leq pB \leq uq. \end{aligned} \tag{40}$$

These are the integer form of Proposition 6.1. The listed thresholds are then checked by multiplying away the five-place mantissa denominator. For example, the two 10000-place records satisfy

$$100000q_{\pi+e} > 197688 \cdot 10^{4997}, \quad 100000q_{\zeta(7)} > 928497 \cdot 10^{4999}.$$

The exact generation rules and all six records are supplied with the programs described in the appendix. Finally, (16) applies to any reduced rational representation of the input. $\square$

These are finite exclusions, supplied as reproducible illustrations of certification. Their size is not evidence that a hypothetical rational denominator is impossible. A rational of the minimum interval denominator actually lies inside each enclosure, so the interval data alone cannot exclude that denominator or every larger denominator. The candidates need not equal the enclosed input.

All computations for Table 1 use integers and rational numbers. The first 24 coefficient identities (32) were also checked against the alternating binomial definition (26). At scales 200 and 1000, an independent exact finite-difference evaluator reproduces the same zeta endpoints. At scale 10000, increasing the working precision by eight guard digits and the transformed cutoff by 50 terms leaves its rounded endpoints unchanged. A separate 10012-place $\pi + e$ enclosure lies strictly inside the reported 10000-place enclosure. These additional calculations check the implementations; the analytic bounds and the integer certificate provide the justification of the exclusions.

8.3. One-million-place records. Apply the certificate of Proposition 6.1 to the three intervals obtained in Section 7.4. Table 2 records their exact widths, minimum-denominator lengths and six-significant-digit downward thresholds. These rows use FLINT/Arb enclosures; they do not extend the defining-series evaluator of Table 1 to a million terms or digits.

Input	D	C_D	Rounded-down threshold	Digits of $q_{\min}$
$\pi + e$	1 000 000	1	$1.49494 \times 10^{500000}$	500 001
$\zeta(7)$	1 000 000	1	$1.66513 \times 10^{500000}$	500 001
γ	1 000 000	1	$2.20474 \times 10^{500000}$	500 001

TABLE 2. FLINT/Arb enclosures rounded outward to the decimal grid. All three widths are exactly $10^{-1\,000\,000}$. The exact interval minima are stored in full in the supplementary records; each displayed threshold is strictly smaller than its corresponding minimum.

Theorem 8.2. *The million-place enclosures and certificates give the following exclusions. Every reduced rational representation of $\pi + e$, if one exists, has denominator*

$$q > 1.49494 \times 10^{500000}.$$

Every reduced rational representation of $\zeta(7)$, if one exists, has denominator

$$q > 1.66513 \times 10^{500000}.$$

Every reduced rational representation of γ , if one exists, has denominator

$$q > 2.20474 \times 10^{500000}.$$

For each enclosure the exact minimum denominator has 500 001 digits.

Proof. Arb's enclosure guarantee places each constant in its recorded dyadic ball. Proposition 7.3 and the strict checks (39) place each ball inside the saved rational interval. The records also satisfy every inequality in (40). Proposition 6.1 therefore gives the exact interval minima

$$Q_\pi = b_\pi + d_\pi, \quad Q_7 = b_7 + d_7, \quad Q_\gamma = b_\gamma + d_\gamma.$$

Their complete decimal expansions have 500 001 digits. The leading thirty digits are

$$\begin{aligned} Q_\pi : & \quad 149494574866266435105686656176, \\ Q_7 : & \quad 166513727099889707103223705527, \\ Q_\gamma : & \quad 220474183942374508487867519207. \end{aligned}$$

Integer comparisons give

$$\begin{aligned} 100000Q_\pi &> 149494 \cdot 10^{500000}, \\ 100000Q_7 &> 166513 \cdot 10^{500000}, \\ 100000Q_\gamma &> 220474 \cdot 10^{500000}. \end{aligned}$$

Any reduced rational representation of the enclosed constant has denominator at least the interval minimum by (16), proving the strict displayed exclusions. $\square$

For this larger computation, FLINT's public `fmpq_simplest_between` routine supplies a reduced candidate p/q in the closed interval. Its determinant-one parents can be recovered without retaining the search path. For $q > 1$, let b be the inverse of p modulo q , chosen in $\{1, \dots, q-1\}$, and put

$$a = \frac{pb-1}{q}, \quad d = q-b, \quad c = p-a. \quad (41)$$

Then a is an integer, $b, d > 0$, $p = a+c$, $q = b+d$ and $bc-ad = pb-aq = 1$. Once the strict outer-endpoint and candidate-containment checks in (40) pass, these identities certify minimality independently of how the candidate was found. A different interval may require a different candidate or an endpoint certificate; the present three records satisfy this determinant-one form.

The reruns with 96 guard digits give the same decimal endpoints, with each finer ball contained in the corresponding primary ball. The search bridge was also checked against exhaustive denominator search on 1500 random closed rational intervals, including negative endpoints and zero-width intervals, and the large-integer path was checked at 100000 places. A million-place evaluation of

$$16 \arctan(1/5) - 4 \arctan(1/239) + \exp(1)$$

gives the same $\pi + e$ grid endpoints within the same Arb library. Additional Arb evaluations at 1000 places lie inside the original defining-series enclosures. These are implementation checks, not independent proofs of the library's constant-evaluation algorithms. No separate million-place zeta evaluator is claimed.

For γ , a separate 200-place check uses exact rational Euler–Maclaurin coefficients and Arb only for $\log N$ and directed rounding. Write

$$S_{N,m} = H_N - \log N - \frac{1}{2N} + \sum_{k=1}^m \frac{B_{2k}}{2kN^{2k}}, \quad T_{N,m} = \frac{B_{2m+2}}{(2m+2)N^{2m+2}},$$

where B_j are Bernoulli numbers. The identity $\psi(N) = H_{N-1} - \gamma$ and the classical positive-real remainder bound for the digamma expansion give, for $N = m = 128$,

$$0 \leq \gamma - S_{128,128} \leq T_{128,128}, \quad 0 < T_{128,128} < 10^{-200}$$

[10, eqs. (5.4.14), (5.11.2) and Section 5.11(ii)]. The program evaluates the rational terms exactly, encloses the logarithm, and records an outward 200-place interval. Exact dyadic comparisons place the entire million-place primary ball strictly inside this independently formed enclosure. This checks a different formula at 200 places; it is not a separate million-place evaluator and still uses Arb for the logarithm.

The supplementary audit uses exact integer arithmetic to check the saved balls, grid endpoints, adjacent fractions, candidates, widths, digit counts and strict thresholds. It does not rerun Arb evaluation or the search. Thus the ball-to-grid and minimum-denominator conclusions are independently checked from the stored integers; containment of the defining constants in those balls relies on the documented numerical library. As before, a rational candidate attaining the interval minimum lies inside each interval, and is not identified with the input.

9. WHY APPROXIMATELY HALF THE PRECISION CAN APPEAR

Write the actual enclosure width as

$$w = U - L = C_D 10^{-D}.$$

For adjacent fractions of determinant one,

$$\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}. \quad (42)$$

If b and d are comparable to a common scale Q , this surrounding gap has size approximately Q^{-2} . This suggests $q_{\min}$ on the order of $w^{-1/2}$. It is a scale argument: a target interval can occupy an arbitrarily small part of the surrounding gap, and the adjacent denominators need not be comparable.

9.1. The continued-fraction error identity. For an irrational real x , define its complete quotients by

$$x_0 = x, \quad a_n = \lfloor x_n \rfloor, \quad x_{n+1} = (x_n - a_n)^{-1}.$$

The convergents $p_n/q_n = [a_0; \dots, a_n]$ have initial values

$$p_{-2} = 0, \quad p_{-1} = 1, \quad q_{-2} = 1, \quad q_{-1} = 0$$

and recurrences $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$. Induction gives

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}, \quad x = \frac{p_n x_{n+1} + p_{n-1}}{q_n x_{n+1} + q_{n-1}}.$$

Subtracting p_n/q_n yields the exact identity

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n^2 (x_{n+1} + q_{n-1}/q_n)}. \quad (43)$$

These classical recurrences [7, 10] explain the inverse-square scale and the fluctuations caused by large complete quotients. Convergents also accelerate rational detection under exact arithmetic: for a rational complete quotient P/Q , its next denominator is the Euclidean remainder, strictly smaller than Q . The expansion therefore terminates. An exact repetition of a complete quotient along nonzero remainders certifies an infinite expansion and hence an irrational input. A repeated finite digit block in a decimal calculation is not such an exact cycle certificate.

9.2. A theorem under explicit approximation hypotheses. A real x is called badly approximable here if there is a constant $c > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^2} \quad \text{for every reduced rational } p/q. \quad (44)$$

This condition implies that x is irrational. An enclosure is well centered for the following result if x stays a fixed fraction of its width away from both endpoints.

Proposition 9.1. *Suppose (44) holds, $L < x < U$, and*

$$\min(x - L, U - x) \geq \theta w, \quad w = U - L, \quad \theta > 0.$$

Then

$$\sqrt{c/w} \leq q_{\min}([L, U]) \leq Q, \quad Q = \left\lceil \frac{1}{\sqrt{c\theta w}} \right\rceil. \quad (45)$$

Here Q is the displayed quantity rounded upward to an integer.

Proof. Any rational in the interval is at distance at most w from x , so (44) gives the lower bound.

For the upper bound, let $Q = \lceil 1/\sqrt{c\theta w} \rceil$. Among the $Q+1$ fractional parts of $0, x, \dots, Qx$, two belong to the same one of Q half-open equal subintervals of $[0, 1)$. Their difference gives integers p, q with $1 \leq q \leq Q$ and $|x - p/q| < 1/(qQ)$. Reducing the fraction, if necessary, preserves the weaker inequality with its reduced denominator, which we again call q .

Combining this estimate with (44) gives $q > cQ$, and hence

$$|x - p/q| < \frac{1}{cQ^2} \leq \theta w.$$

This rational belongs to $[L, U]$ and has denominator at most Q , proving the upper bound. $\square$

For intervals with a common centering constant θ and widths tending to zero, the proposition gives

$$\log_{10} q_{\min} = \frac{D - \log_{10} C_D}{2} + O(1). \quad (46)$$

Neither the approximation hypothesis nor a uniform centering hypothesis is established here for $\pi + e$, $\zeta(7)$ or γ . Their rows in Tables 1 and 2 are exact individual calculations, not applications of Proposition 9.1. At the million-place scale, $C_D = 1$ and each computed minimum has 500 001 digits, so its scientific-notation exponent is $500000 = D/2$. This observation establishes the half-scale pattern for these three intervals only; it supplies no universal rate or rationality classification.

An elementary example of (44) is $x = \sqrt{2}$, with $c = 1/4$. If $|\sqrt{2} - p/q| \leq 1$, then

$$|\sqrt{2} - p/q| = \frac{|2q^2 - p^2|}{q^2|\sqrt{2} + p/q|} > \frac{1}{4q^2},$$

because the nonzero integer numerator has magnitude at least one and the second factor is at most $2\sqrt{2} + 1 < 4$. If the distance exceeds one, the same lower bound is immediate.

9.3. Counterexamples to a universal width law. First, an interval of width 10^{-D} centered at $1/2$ has $q_{\min} = 2$ for every $D \geq 1$. It also contains irrational points, for example $1/2 + \sqrt{2}10^{-D-2}$. High precision alone therefore does not force a large minimum denominator.

At the opposite extreme, let $D \geq 3$, $B = 10^D$ and

$$I_D = [1/2 + 1/B, 1/2 + 2/B]. \quad (47)$$

Then

$$q_{\min}(I_D) = B/4 + 1. \quad (48)$$

Indeed, a positive difference $p/q - 1/2$ is at least $1/(2q)$, excluding all $q < B/4$. At $q = B/4$, the denominator is even, so the positive numerator $2p - q$ is at least two and the difference is at least $1/q = 4/B$, also outside the interval. At the next denominator $q = B/4 + 1$, which is odd, the reduced fraction $(q+1)/(2q)$ differs from $1/2$ by $2/(B+4) \in [1/B, 2/B]$. Thus a width 10^{-D} can instead have a minimum denominator of order 10^D .

There is also a counterexample with one fixed irrational input. Put

$$x = \sum_{k \geq 1} 10^{-k!}, \quad s_n = \sum_{k=1}^n 10^{-k!}, \quad J_n = [s_n, s_n + 2 \cdot 10^{-(n+1)!}] \quad (n \geq 2).$$

Successive terms have ratios at most $1/10$, so $0 < x - s_n < 2 \cdot 10^{-(n+1)!}$ and $x \in J_n$. To prove irrationality directly, suppose $x = p/q$. Since $10^{n!} s_n$ is an integer, the strictly positive number

$$q10^{n!}(x - s_n)$$

would be an integer; its upper bound $2q10^{n!-(n+1)!}$ tends to zero, a contradiction. On the other hand, $s_n \in J_n$ has reduced denominator at most $10^{n!}$, so

$$q_{\min}(J_n)\sqrt{|J_n|} \leq \sqrt{2}10^{n!-(n+1)!/2} \longrightarrow 0.$$

The same tail-ratio estimate shows that $J_{n+1} \subset J_n$. Even for a fixed irrational input, the inverse-square-root scale is not universal. Proposition 6.4 still forces unbounded minimum denominators along these nested intervals; it specifies no universal rate.

10. THE RELATION BETWEEN CHARACTERIZATION AND CERTIFICATION

The exact hit test $\tau(qx)$ and the interval count $H_I(q)$ answer different arithmetic questions. An integer hit identifies a rational input and a denominator; a positive interval count identifies a rational candidate compatible with the enclosure. The latter does not identify the candidate with the input.

The positive telescoping triple characterizes rationality by equality of its totals while preserving positivity and order of every partial sum. Its compensating reversal is compatible with Theorem 4.4, which excludes complete termwise ordering within the stated class of all-real finite expressions. A defining integer polynomial permits ordered certificates. Arbitrary approximation programs, in contrast, are subject to the finite-decision obstruction of Proposition 5.1.

Certified intervals yield finite denominator exclusions through exact adjacent-fraction identities. Section 8.1 gives the complete calculation at a small scale; the larger records apply the same certificate. Their interval minima are verified by integer arithmetic, while the million-place constant enclosures also use Arb's rigorous evaluation guarantee. Proposition 9.1 explains the inverse-square-root scale under approximation and centering hypotheses; the counterexamples show why those hypotheses cannot be discarded. None of the finite computations determines the rationality of the application constants.

APPENDIX A. REPRODUCTION AND EXACT CERTIFICATE CHECKS

A.1. Defining-series computations. The defining-series portion of the supplementary package contains five standard-library Python programs and six exact integer records. From its root directory, the complete calculation is reproduced by

```
python3 code/reproduce_certificates.py
--digits 200 1000 10000 --output results
```

The displayed lines form one command. The programs use integer division for floors and ceilings and the Python `Fraction` type for rational endpoints. Decimal formatting is used only for descriptive output.

For an independent check that does not execute the search, run

```
python3 code/audit_certificates.py --results certificates
```

This checks (40), the exact widths and denominator lengths, and all strict thresholds in Table 1. The exact minimum is determined by the certificate rather than by a claim that the search checked every smaller denominator. A separate program, `check_constructions.py`, checks rational examples, inverse counts, small-interval searches against exhaustive enumeration, and rounded recurrence bounds against exact coefficients. These finite checks supplement the proofs; they do not replace the quantifiers in the theorems.

The recorded truncation parameters for $\pi + e$ are

D	K_5	K_{239}	M
200	143	42	123
1000	715	210	452
10000	7153	2102	3250

The corresponding zeta cutoffs, with 12 guard digits, are $K = 705, 3362$ and 33260 . The exact records also retain the adjacent fractions and compressed jump counts. The two additional precision checks described in Section 8 are available as an optional audit command. Every substantive implementation used for the numerical claims is included in the supplementary source.

A.2. Million-place computations. For the million-place saved records, a standalone standard-library checker is also supplied:

```
python3 code/audit_million_certificates.py
--results million/results
```

The two lines form one command. This checker imports neither the numerical evaluator nor the native search bridge and needs no FLINT, Arb or GMP installation. It verifies the same determinant and containment certificate, the exact dyadic floors and ceilings, digit counts and strict downward thresholds using Python integers. It does not independently prove the library’s constant-enclosure algorithms.

The separate `million/` directory contains seven Python programs, the three full million-place certificates, summary files and captured validation logs. The tested environment was Python 3.12.14, `python-flint` 0.9.0 with FLINT 3.6.0, and `gmpy2` 2.3.2 with GMP 6.3.0, on LP64 Linux, using four FLINT threads. These computations require the external numerical libraries, which are not bundled. From the supplementary root, install the Python dependencies with

```
python3 -m pip install python-flint==0.9.0 gmpy2==2.3.2
```

and reproduce all three enclosures, candidate searches and guard-digit reruns with

```
python3 million/certified_million.py
--digits 1000000 --output million/reproduced
```

The two displayed lines form one command. To check the supplied records without rerunning the numerical evaluations or the search, use

```
python3 million/verify_certificates.py million/results
```

This integer audit imports the evaluation modules, and therefore requires their dependencies, but calls neither Arb evaluation nor the native fraction search. The native bridge calls FLINT’s public C API through owned rational objects; it requires an LP64 build and rejects an incompatible pointer/long layout. Its candidate output is certified by (40), rather than by trusting its search claim.

The additional interval and expression checks described in Section 8 are reproduced by the two separate commands

```
python3 million/check_native_interval.py
python3 million/alternate_check.py
```

The latter compares a different expression for $\pi + e$ within Arb and checks the original 1000-place enclosures; it does not supply an independent million-place zeta algorithm. The gamma formula check is reproduced by

```
python3 million/check_gamma_euler_maclaurin.py
```

Captured logs preserve the original runs and the added gamma computation. Package manifests record the exact sources and data. The accompanying `Supplementary_Material.zip` contains both tiers and the small example; its `Verification_Guide.md` identifies the files, commands, dependencies and expected conclusions. The worked example is reproduced by

```
python3 code/worked_example.py --output examples
```

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