

FLOOR–CEILING ROOT ZETA FUNCTIONS: ZERO GEOMETRY AND MELLIN CRITERIA

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ABSTRACT. Integer-root floor and ceiling zeta functions are finite linear combinations of shifted Riemann zeta functions. For each fixed root order $q \geq 2$ and all sufficiently large n , each disk of radius $1/2$ centred at $q - 1 - 2n$ contains a unique simple zero of each family, which is real. We compute the displacement through order n^{-3} , with no n^{-2} term, and obtain paired separation and midpoint asymptotics. We also construct a Möbius-weighted L -point interpolation with a holomorphic rounding error on $\operatorname{Re} s > q - L$. At $L = q$, every nontrivial reciprocal-zeta pole is transported with its multiplicity. A regularized geometric kernel connects these constructions to the classical Riesz framework: its Mellin numerator cancels no nontrivial Riemann zero, and its optimal decay exponent characterizes their rightmost real part. We give an explicit complete-tail acceleration formula for evaluating this kernel and supporting finite Gram, smoothed-totient and reciprocal-Fibonacci identities. Local complex-zero experiments complement the analytic results. The RH-equivalent decay and arithmetic estimates remain unproved.

1. INTRODUCTION

Rounded index maps replace a sum over repeated values by a sum over occupied levels. For integer root order q , the positive-level floor and ceiling multiplicities are polynomial:

$$f_q(j) = (j + 1)^q - j^q, \quad c_q(j) = j^q - (j - 1)^q.$$

The resulting Dirichlet transforms, denoted by F_q and C_q , reduce to finite combinations of shifted zeta functions [1, 2]. Their zeros need not be zeros of any individual summand. This distinction is important when interpreting numerical curves near trivial or nontrivial Riemann-zero heights.

1.1. Principal results. The first main result is an analytic description of the paired far-left zeros. Fix $q \geq 2$ and put $d = q - 1$. Theorem 10.1 proves that, for every sufficiently large n , each of F_q and C_q has exactly one zero in $|s - (d - 2n)| < 1/2$. The zero is simple and real, and

$$s_{q,n}^\sigma = d - 2n + \frac{\sigma d}{n} + \frac{d^2/2 + \sigma \pi^2 d(d - 2)/12}{n^3} + O_q(n^{-4}),$$

where $\sigma = -1$ for the floor family and $\sigma = +1$ for the ceiling family. In particular, the quadratic term vanishes. Corollary 10.2 gives the separation of the two zeros and the displacement of their midpoint. This turns a reciprocal bending pattern into a quantified zero law for each fixed q .

The second main result makes the effect of rounding on reciprocal-zeta poles explicit. Theorem 6.1 constructs a Möbius-weighted L -point interpolation $U_{q,L}$ with

$$U_{q,L}(s) = \frac{1}{\zeta(s/q)} + E_{q,L}(s), \quad E_{q,L} \text{ holomorphic on } \operatorname{Re} s > q - L.$$

The choice $L = q$ transports every nontrivial Riemann zero ρ to an uncanceled pole at $q\rho$, with the same multiplicity. Polynomial reproduction supplies the interpolation, while a normally convergent error series gives the stated continuation domain. Thus the pole

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statement concerns all nontrivial zeros and does not depend on numerical continuation from selected seeds.

These two results describe different analytic roles of rounded index maps. The first locates a zero family of the unweighted transforms; the second preserves the poles of a reciprocal-zeta function despite rounding. A regularized kernel connects the root-count identities with a third, classical route to Riemann-zero information. The exact cancellations

$$F_2(s) - C_2(s) = 2\zeta(s), \quad F_3(s) - C_3(s) = 6\zeta(s-1)$$

lead to

$$d(t) = \frac{1}{t^2} - \frac{e^t}{(e^t - 1)^2}, \quad \mathcal{R}_d(x) = \sum_{n \geq 1} \frac{\mu(n)}{n^2} d(x/n^2).$$

Its Mellin transform is

$$-\frac{\Gamma(s)\zeta(s-1)}{\zeta(2-2s)} \quad (0 < \operatorname{Re} s < 1/2).$$

Theorem 3.3 proves that every nontrivial zero ρ gives an uncanceled pole at $1 - \rho/2$, with its multiplicity. Let

$$\Theta = \sup_{\zeta(\rho)=0, 0 < \operatorname{Re} \rho < 1} \operatorname{Re} \rho, \\ A_d = \sup\{a : \mathcal{R}_d(x) = O_\epsilon(x^{-a+\epsilon}) \text{ for every } \epsilon > 0\}.$$

Theorem 4.4 then proves

$$A_d = 1 - \Theta/2, \quad 1/2 \leq A_d \leq 3/4.$$

The critical decay condition is equivalent to RH. The unconditional estimate proved here is $O(x^{-1/2})$; the identity characterizes the location parameter without determining its value.

1.2. Comparison with prior work. The finite shifted-zeta representations and initial root examples are the starting point supplied by [1, 2]. The specific addition in Theorem 10.1 is the uniform disk argument for each fixed q , followed by the cubic displacement, the missing quadratic term and the paired asymptotics. Local implicit differentiation and square-root splitting are established tools; the zero-motion study [4] illustrates the parameter-differentiation approach. Their application here supplies explicit coefficients for the root-count deformations.

In Theorem 6.1, Lagrange interpolation and the reciprocal-zeta Möbius series are classical ingredients. The contribution is their explicit combination on rounded root blocks, with a quantified holomorphic error on $\operatorname{Re} s > q - L$. Choosing $L = q$ makes that error holomorphic across the entire nontrivial-zero strip after scaling, so that it cannot remove a reciprocal-zeta pole. The finite block coefficients and the explicit remainder bound also make the construction directly evaluable.

The exponent $-3/4 + \epsilon$ belongs to the classical Riesz criterion. Agarwal, Garg and Maji [9] recover that criterion and prove a family of related Möbius-decay equivalences. Here the useful specialization is the elementary positive kernel d , its numerator $\Gamma(s)\zeta(s-1)$ that is nonzero at every mapped nontrivial zero, and its controlled evaluation. Lemma 3.1 gives elementary endpoint bounds, and Theorem 5.1 sums the complete tail through reciprocal-zeta moments with an explicit analytic remainder. The decay-frontier theorem records exactly which zero-location parameter this kernel encodes; it does not improve the unconditional location of Riemann zeros.

The remaining constructions support these analytic results. Báez-Duarte's integer-scale approximation criterion [6], its Dirichlet-polynomial analysis [8], and Gram-matrix formulas such as those in [7] provide the established Hilbert-space setting. Section 9 supplies a finite residue formula for the present floor-difference basis and a dyadic obstruction. Smoothed totient errors are already linked to zeta zeros in [10]; Section 7 supplies exact polynomial cutoff identities and a balanced combination preserving all nontrivial poles. The reciprocal-Fibonacci family in Section 8 illustrates how sufficiently large shifts prevent simultaneous cancellation

at any fixed zero. Rational-scale models in Section 11 connect with the Hurwitz-combination setting of [5].

1.3. Organization and scope. The two principal proofs can be read separately: Section 2 supplies the shifted-zeta formulas needed for the far-left theorem in Section 10, while Section 6 proves the rounded interpolation theorem directly. Sections 3–5 develop the regularized kernel, its decay frontier and controlled evaluation. Sections 7–9 give the supporting arithmetic families and approximation formulas. The rest of Section 10 and Section 11 concern local complex-zero geometry and parameter models. Section 12 records the numerical evidence and the remaining arithmetic estimates.

The paper proves specified rounded-zeta zero asymptotics and explicit constructions preserving reciprocal-zeta poles. The critical kernel decay, vanishing approximation error and square-root smoothed-totient error remain unproved. No new unconditional fixed strip for all Riemann zeros, or proof of RH, is asserted.

1.4. Conventions. The root order q is a positive integer unless stated otherwise. The symbols μ , Γ and ζ have their usual meanings. Mellin integrals are ordinary convergent integrals on their stated strips. A continued identity is distinguished from its defining Dirichlet series. Constants in O_q estimates may depend on fixed q . Numerical root paths are not treated as certified global zero counts.

2. FLOOR AND CEILING ROOT ZETA FUNCTIONS

Fix $q \in \mathbb{N}$ and put $d = q - 1$. The full positive-level counts are

$$f_q(j) = (j + 1)^q - j^q, \quad c_q(j) = j^q - (j - 1)^q \quad (j \geq 1).$$

For $\operatorname{Re} s > q$, define

$$(2.1) \quad F_q(s) = \sum_{j \geq 1} f_q(j) j^{-s}, \quad C_q(s) = \sum_{j \geq 1} c_q(j) j^{-s}.$$

The unit-scale floor map on positive integer inputs has no zero-level contribution. These are exactly the unshifted root-floor and root-ceiling series of [2, Corollaries 3–4].

Proposition 2.1. *The meromorphic continuations are*

$$(2.2) \quad F_q(s) = \sum_{k=0}^{q-1} \binom{q}{k} \zeta(s - k),$$

$$(2.3) \quad C_q(s) = \sum_{k=0}^{q-1} (-1)^{q-k+1} \binom{q}{k} \zeta(s - k).$$

Every integer $1, \dots, q$ is a simple pole. The residues of F_q are positive; those of C_q alternate. Both functions respect complex conjugation.

Proof. Expand each count by the binomial theorem in the domain of absolute convergence. Each shifted zeta function has its only pole at $s = k + 1$, so no second summand can cancel that pole. Real coefficients give conjugation symmetry. Analytic uniqueness supplies the continuation. $\square$

For example,

$$\begin{aligned} F_2 &= 2\zeta(s - 1) + \zeta(s), \\ C_2 &= 2\zeta(s - 1) - \zeta(s), \\ F_3 &= 3\zeta(s - 2) + 3\zeta(s - 1) + \zeta(s), \\ C_3 &= 3\zeta(s - 2) - 3\zeta(s - 1) + \zeta(s), \\ F_4 &= 4\zeta(s - 3) + 6\zeta(s - 2) + 4\zeta(s - 1) + \zeta(s), \\ C_4 &= 4\zeta(s - 3) - 6\zeta(s - 2) + 4\zeta(s - 1) - \zeta(s). \end{aligned}$$

The s argument on the left is suppressed only in this display. Equations (2.2)–(2.3) concern continuation; their defining Dirichlet series do not converge throughout the displayed zero regions.

2.1. Zeros are not inherited term by term. At a zero of one summand, the remaining summands need not vanish. In particular,

$$(2.4) \quad F_2(-2) = \frac{1}{60}, \quad C_2(-2) = \frac{1}{60}, \quad F_2(0) = -\frac{2}{3}, \quad C_2(0) = \frac{1}{3}.$$

These follow from the standard integer zeta values [3].

Proposition 2.2. *If ρ is any nontrivial Riemann zero, neither ρ nor $1 + \rho$ is a zero of F_2 or C_2 . Furthermore F_2 and C_2 have no common zero.*

Proof. At $s = \rho$, the value is $2\zeta(\rho - 1)$. The real part of $\rho - 1$ lies in $(-1, 0)$ and its imaginary part is nonzero, so this value is nonzero by the standard location of Riemann zeros. At $s = 1 + \rho$, the values are respectively $\zeta(1 + \rho)$ and $-\zeta(1 + \rho)$, nonzero in $\operatorname{Re} s > 1$. A common zero would require $\zeta(s) = \zeta(s - 1) = 0$. Their nontrivial-zero strips are disjoint, their negative even zeros cannot differ by one, and a real trivial zero cannot be paired with a nonreal zero. These exhaust the possibilities [3]. $\square$

This assertion uses no Riemann hypothesis. The computations below use the first six known critical-line zeros as specific seeds.

2.2. Floor–ceiling cancellation supplies exact comparison zeros. Adding or subtracting (2.2)–(2.3) selects a parity of shifts:

$$(2.5) \quad F_q + C_q = 2 \sum_{\substack{0 \leq k < q \\ q-k \text{ odd}}} \binom{q}{k} \zeta(s - k),$$

$$(2.6) \quad F_q - C_q = 2 \sum_{\substack{0 \leq k < q \\ q-k \text{ even}}} \binom{q}{k} \zeta(s - k).$$

In particular,

$$(2.7) \quad F_2 - C_2 = 2\zeta(s), \quad F_3 - C_3 = 6\zeta(s - 1).$$

Thus the original functions themselves produce exact Riemann-zero comparison sets. A zero of the difference means equality of two values, not their separate vanishing. These identities also provide sensitive implementation checks.

2.3. Zeros in the convergence half-plane and a far-right bound. Positive Dirichlet coefficients exclude zeros on the real axis of absolute convergence, but do not exclude complex cancellation. In particular, the first computed C_2 zero near a Riemann height has real part $2.046038604155 > 2$. It lies in the convergence half-plane of the original ceiling series. It is a zero of a sum of shifts, not an off-strip zero of the Riemann zeta function.

Proposition 2.3. *Put $\sigma = \operatorname{Re} s > q$. The floor function is nonzero if $\zeta(\sigma - q + 1) - 1 < 1$. The ceiling function is nonzero if $q(\zeta(\sigma - q + 1) - 1) < 1$.*

Proof. The first coefficients are $f_q(1) = 2^q - 1$ and $c_q(1) = 1$. For $j \geq 1$, $f_q(j) \leq (2^q - 1)j^{q-1}$ by the positive binomial coefficients, and $c_q(j) \leq qj^{q-1}$ by integration of qx^{q-1} from $j - 1$ to j . The absolute modulus of the remaining Dirichlet-series tail is then bounded by $(2^q - 1)(\zeta(\sigma - q + 1) - 1)$ or $q(\zeta(\sigma - q + 1) - 1)$, respectively. If it is smaller than the first coefficient, that coefficient cannot be cancelled. $\square$

These are sufficient bounds, not sharp zero-free boundaries. They explain why roots in $\operatorname{Re} s > q$ can occur without implying zeros arbitrarily far to the right.

2.4. Positive real zeros.

Theorem 2.4. *For every $q \geq 2$ and $k = 1, \dots, q-1$, F_q has at least one real zero in $(k, k+1)$. It has no real zero in (q, ∞) .*

Proof. The positive residue at k gives $F_q(s) \rightarrow +\infty$ as $s \downarrow k$, and the positive residue at $k+1$ gives $F_q(s) \rightarrow -\infty$ as $s \uparrow k+1$. Continuity on the interval proves existence. For real $s > q$, every term of (2.1) is positive. $\square$

This explains the positive roots printed in [2] without classifying them as shifted Riemann roots. The argument proves a lower bound of $q-1$, not an upper bound or uniqueness in each interval. Alternating residues give C_q the same sign at the two ends of each such interval, so the analogous existence argument does not apply. Positivity excludes real C_q zeros for $s > q$, but does not exclude complex zeros there.

3. THE REGULARIZED CUBIC-ROOT MELLIN KERNEL

3.1. The original geometric functions. Set $A(t) = (e^t - 1)^{-1}$ for $t > 0$. Positive-level root-count traces are

$$h_{q,F}(t) = \sum_{j \geq 1} f_q(j) e^{-jt}, \quad h_{q,C}(t) = \sum_{j \geq 1} c_q(j) e^{-jt}.$$

These are the geometric functions of Part II with the floor zero-level term deleted. Their Mellin transforms are $\Gamma(s)F_q(s)$ and $\Gamma(s)C_q(s)$ in $\text{Re } s > q$.

For $q = 2$,

$$h_{2,F} = -2A' + A, \quad h_{2,C} = -2A' - A.$$

Their small- t singular terms are $2/t^2 + 1/t$ and $2/t^2 - 1/t$, respectively. Define

$$b(t) = A(t) - \frac{1}{t}, \quad d(t) = b'(t) = \frac{1}{t^2} - \frac{e^t}{(e^t - 1)^2}.$$

After subtracting the singular powers,

$$(3.1) \quad \hat{h}_{2,F} = b - 2d, \quad \hat{h}_{2,C} = -b - 2d.$$

The counterterms are subtracted explicitly; all Mellin integrals below converge ordinarily on the specified strips.

The cubic-root counts give a particularly useful cancellation:

$$(3.2) \quad F_3 - C_3 = 6\zeta(s-1), \quad h_{3,F} - h_{3,C} = 6 \frac{e^t}{(e^t - 1)^2}.$$

Consequently,

$$(h_{3,F} - h_{3,C}) - \frac{6}{t^2} = -6d(t).$$

The cube-root floor-ceiling difference therefore produces a regularized positive kernel whose zeta numerator is shifted to the zero-free left half-plane.

Lemma 3.1. *For $t > 0$,*

$$-\frac{1}{2} < b(t) < 0, \quad 0 < d(t) \leq \min \left\{ \frac{1}{12}, \frac{1}{t^2} \right\}.$$

The limits at zero are $b(0) = -1/2$ and $d(0) = 1/12$.

Proof. Write $b(t) = \frac{1}{2} \coth(t/2) - \frac{1}{2} - 1/t$. Its derivative is positive because $2 \sinh(t/2) > t$. Its zero and infinity limits give the bound on b . The formula for d gives $d \leq t^{-2}$. The classical partial fraction expansion of $\coth$ [3, Section 4.36] gives

$$d(t) = 2 \sum_{\ell \geq 1} \frac{(2\pi\ell)^2 - t^2}{((2\pi\ell)^2 + t^2)^2}.$$

Each term is at most $2/(2\pi\ell)^2$. Summation gives $d \leq 1/12$. The expansion at zero gives the two limits. $\square$

Proposition 3.2. *The ordinary regularized Mellin integrals satisfy*

$$(3.3) \quad \int_0^\infty b(t)t^{s-1} dt = \Gamma(s)\zeta(s), \quad 0 < \operatorname{Re} s < 1,$$

$$(3.4) \quad \int_0^\infty d(t)t^{s-1} dt = -\Gamma(s)\zeta(s-1), \quad 0 < \operatorname{Re} s < 2.$$

The regularized pair in (3.1) has Mellin transforms $\Gamma(s)F_2(s)$ and $\Gamma(s)C_2(s)$ on $0 < \operatorname{Re} s < 1$.

Proof. Start with the classical exponentially convergent integrals in their original half-planes [3, Section 25.5]. Split at $t = 1$, subtract t^{-1} from A or t^{-2} from $-A'$, and retain the elementary integrated counterterms. For example, the continued expression for the first transform is

$$\int_0^1 b(t)t^{s-1} dt + \int_1^\infty A(t)t^{s-1} dt + \frac{1}{s-1}.$$

On $0 < \operatorname{Re} s < 1$ it is exactly the integral of b over the full positive axis. For the second, the original transform of A' is $-\Gamma(s)\zeta(s-1)$. Subtracting its singular term $-t^{-2}$ and adding it back as $-(s-2)^{-1}$ gives the integral of d on $0 < \operatorname{Re} s < 2$. Bounds at zero and infinity justify ordinary convergence in the stated strips. Equation (3.1) then proves the assertion for the pair. $\square$

3.2. Möbius weighting turns the numerator into a zero detector. Let μ be the Möbius function and define

$$(3.5) \quad \mathcal{R}_d(x) = \sum_{n \geq 1} \frac{\mu(n)}{n^2} d\left(\frac{x}{n^2}\right), \quad x > 0.$$

This is an ordinary absolutely convergent series, locally uniformly in x , by Lemma 3.1. Absolute convergence permits the weighted substitutions used below.

Theorem 3.3. *On $0 < \operatorname{Re} s < 1/2$,*

$$(3.6) \quad \int_0^\infty \mathcal{R}_d(x)x^{s-1} dx = -\frac{\Gamma(s)\zeta(s-1)}{\zeta(2-2s)}.$$

Every nontrivial Riemann zero ρ , of any multiplicity, produces an uncanceled pole at $s_\rho = 1 - \rho/2$ in the meromorphic continuation of the right side. Its order equals the multiplicity of ρ . At a simple zero the residue is

$$\frac{\Gamma(s_\rho)\zeta(s_\rho-1)}{2\zeta'(\rho)}.$$

Proof. Absolute integration and the substitution $x = n^2t$ give (3.6), using $\sum \mu(n)n^{-w} = 1/\zeta(w)$ on $\operatorname{Re} w > 1$ [3, Section 27.4]. For a nontrivial zero, $0 < \operatorname{Re} \rho < 1$ and $\operatorname{Im} \rho \neq 0$. Thus $s_\rho - 1 = -\rho/2$ has real part in $(-1/2, 0)$ and nonzero imaginary part. The zeta function is nonzero there; its only zeros in the negative-real half-plane are negative even real numbers. Gamma has no zeros or poles at s_ρ . The denominator zero is therefore never canceled. Differentiating its argument gives the residue, including its sign. $\square$

This covers all nontrivial Riemann zeros, without a completeness assumption about a numerical root list. It is stronger as a detector than a single multiplier whose complex zeros have not been controlled.

4. DECAY EXPONENTS AND GLOBAL ZERO LOCATION

Theorem 4.1. *Let $1/2 < a \leq 3/4$. If, for every $\epsilon > 0$,*

$$\mathcal{R}_d(x) = O_\epsilon(x^{-a+\epsilon}) \quad (x \rightarrow \infty),$$

then every nontrivial Riemann zero satisfies

$$(4.1) \quad 2a - 1 \leq \operatorname{Re} \rho \leq 2 - 2a.$$

In particular the estimate with $a = 3/4$ implies RH. The critical estimate is not assumed.

Proof. The defining series is bounded at zero. The assumed decay makes its Mellin transform holomorphic on $0 < \operatorname{Re} s < a$. Identity (3.6) and analytic uniqueness identify that holomorphic function with the continued quotient there. If $\operatorname{Re} \rho > 2 - 2a$, its mapped pole has $\operatorname{Re} s_\rho < a$, contradicting Theorem 3.3. This proves the upper bound; the functional equation supplies the lower one. $\square$

Theorem 4.2. *RH is equivalent to*

$$\mathcal{R}_d(x) = O_\epsilon(x^{-3/4+\epsilon}) \quad \text{for every } \epsilon > 0.$$

This is a specialized Riesz-type criterion [9].

Proof. The converse is Theorem 4.1. Assume RH for the forward direction only. In every closed half-plane $\operatorname{Re} w \geq 1/2 + \delta$, the standard RH-conditional reciprocal bounds give polynomial, indeed subpolynomial, growth for $1/\zeta(w)$. One precise input is the zero-free-half-plane estimate quoted in [6, Lemma 2.1]: taking its finite cutoff equal to two bounds $1/\zeta(w)$ by $O_{\delta,\eta}((1 + |\operatorname{Im} w|)^\eta)$ on $1/2 + \delta \leq \operatorname{Re} w \leq 1$. The same lemma bounds all partial sums $\sum_{n \leq Y} \mu(n)n^{-1-it}$ by $O_\eta((1 + |t|)^\eta)$. Abel summation of those bounded partial sums extends this bound uniformly to $\operatorname{Re} w \geq 1$.

Mellin inversion for (3.6) is valid on an initial line $0 < c < 1/2$. Move that line to $a = 3/4 - \delta$. Under RH the quotient has no pole between the two lines; the pole of ζ at denominator argument 1 produces a zero, not a singularity. Stirling's formula gives exponential vertical decay for Gamma, while the numerator zeta and the reciprocal denominator have at most polynomial growth. The horizontal contour integrals tend to zero, and the new vertical integral is absolutely convergent. Its modulus is bounded by a constant times x^{-a} . Taking arbitrary $\delta > 0$ proves the asserted bound. $\square$

Only the forward implication in this equivalence uses RH. The proof direction needed for an attempted RH proof is the unconditional implication from a decay estimate to a zero-free region.

Proposition 4.3. *Unconditionally, $\mathcal{R}_d(x) = O(x^{-1/2})$. More explicitly, if $x \geq 1$ and $Y = \lfloor \sqrt{x} \rfloor$,*

$$|\mathcal{R}_d(x)| \leq \frac{Y(Y+1)(2Y+1)}{6x^2} + \frac{1}{12Y} = \left(\frac{5}{12} + O(x^{-1/2}) \right) x^{-1/2}.$$

No bound $\mathcal{R}_d(x) = O(x^{-a})$ with $a > 3/4$ can hold.

Proof. For $n \leq Y$ use $d(x/n^2) \leq n^4/x^2$, and for $n > Y$ use $d \leq 1/12$. Sum the first bound exactly and compare the remaining reciprocal squares with an integral. For sharpness, use the established existence of a Riemann zero on the critical line [3, Section 25.10]. Its uncanceled mapped pole has real part $3/4$. Faster uniform decay would make the Mellin transform holomorphic across that pole, a contradiction. $\square$

The proved baseline leaves a quarter-power gap to the RH target. Taking absolute values cannot supply that improvement; it removes the Möbius cancellation being sought.

4.1. An exact global relation between zero location and decay. Let

$$\Theta = \sup_{\zeta(\rho)=0, 0 < \operatorname{Re} \rho < 1} \operatorname{Re} \rho, \quad A_d = \sup \left\{ a : \mathcal{R}_d(x) = O_\epsilon(x^{-a+\epsilon}) \text{ for every } \epsilon > 0 \right\}.$$

These definitions concern all nontrivial zeros and the complete infinite kernel.

Theorem 4.4. *Unconditionally,*

$$A_d = 1 - \frac{\Theta}{2}, \quad \frac{1}{2} \leq A_d \leq \frac{3}{4}.$$

Thus the decay frontier is an exact global location parameter. The equality does not determine either unknown parameter.

Proof. The baseline proves $A_d \geq 1/2$, and the critical-line pole proves $A_d \leq 3/4$. Any admissible exponent a forces $\Theta \leq 2 - 2a$ by the pole argument, giving $A_d \leq 1 - \Theta/2$.

If $\Theta = 1$, the baseline gives the reverse inequality immediately. Otherwise fix $1/2 < a < 1 - \Theta/2$. The denominator argument on the line $\operatorname{Re} s = a$ lies in a closed zero-free half-plane strictly to the right of Θ . The zero-free-half-plane reciprocal bound in [6, Lemma 2.1], with $\alpha = \Theta$, gives at most polynomial vertical growth up to $\operatorname{Re} w = 1$. To extend the bound farther right, let $A_t(Y) = \sum_{n \leq Y} \mu(n)n^{-1-it}$. The same lemma bounds $A_t(Y)$ uniformly in Y . For $u > 0$, Abel summation and the Euler-product identity give

$$\frac{1}{\zeta(1+u+it)} = u \int_1^\infty A_t(y)y^{-u-1} dy, \quad \left| \frac{1}{\zeta(1+u+it)} \right| \leq \sup_{y \geq 1} |A_t(y)|.$$

Thus the bound is uniform across the required closed vertical strip. Gamma again gives exponential decay. Shift the Mellin contour to a exactly as in Theorem 4.2. There are no intervening poles by the definition of Θ . This gives $\mathcal{R}_d(x) = O_a(x^{-a})$. Letting a increase to $1 - \Theta/2$ proves the reverse inequality. $\square$

For example, an unconditional decay exponent $a = 5/8$ would force every nontrivial zero into $1/4 \leq \operatorname{Re} \rho \leq 3/4$. The critical exponent $a = 3/4$ collapses that strip to the critical line. Neither improved exponent is established here.

4.2. Abel summation exposes the arithmetic cancellation. Write $M(y) = \sum_{n \leq y} \mu(n)$ and $D(t) = d(t) + td'(t)$. Abel summation gives the exact identity

$$(4.2) \quad \mathcal{R}_d(x) = 2 \int_1^\infty M(y)y^{-3}D(x/y^2) dy.$$

The boundary term at infinity vanishes, since $M(y) = O(y)$ and d is bounded. The lower boundary is taken at 1^- , before the first jump of M .

Proposition 4.5. *If $0 < \theta < 2$ and $|M(y)| \leq Cy^\theta$ for $y \geq 1$, then*

$$|\mathcal{R}_d(x)| \leq Cx^{-1+\theta/2} \int_0^\infty t^{-\theta/2} |D(t)| dt.$$

The integral is finite. In particular a bound $M(y) = O_\epsilon(y^{1/2+\epsilon})$ gives the critical decay target.

Proof. Differentiate the Abel weight $y^{-2}d(x/y^2)$ to obtain $-2y^{-3}D(x/y^2)$. Apply the assumed bound in (4.2) and substitute $t = x/y^2$; enlarging the resulting integration interval to $(0, \infty)$ proves the formula. At zero $D(t) = 1/12 + O(t^2)$; at infinity $D(t) = -t^{-2} + O(te^{-t})$. These estimates prove integrability for $0 < \theta < 2$. $\square$

This is a transfer statement, not a bound on M . The Mertens estimate displayed here is itself a classical RH-equivalent target. Periodic-prefix estimates alone do not supply this Möbius bound.

4.3. What happens to the square-root pair and the simpler geometric kernel. Möbius weighting of $\widehat{h}_{2,F}$ and $\widehat{h}_{2,C}$ gives two transforms $\Gamma(s)F_2(s)/\zeta(2-2s)$ and $\Gamma(s)C_2(s)/\zeta(2-2s)$. Proposition 2.2 guarantees that at least one detects every denominator zero. If both weighted kernels satisfy the decay bound of Theorem 4.1, the same global strip law follows.

For $\mathcal{R}_g(x) = \sum \mu(n)n^{-2}b(x/n^2)$, the Mellin numerator is $\Gamma(s)\zeta(s)$. Even this single kernel gives an RH-equivalent $x^{-3/4+\epsilon}$ condition. If the condition holds and a zero ρ has real part greater than $1/2$, holomorphy forces $\zeta(1-\rho/2) = 0$ to cancel its denominator pole. The affine map $T(z) = 1 - z/2$ preserves $1/2 < \operatorname{Re} z < 1$. Repetition would force zeros at every $T^k(\rho)$, approaching $2/3$. Zeta is holomorphic and nonzero at the real point $2/3$, so this accumulation is impossible. The forward implication follows by the same RH-conditional contour shift. This contraction argument explains why isolated numerator cancellation cannot hide every off-line zero. The derivative detector $\mathcal{R}_d$ avoids the cancellation at the first step.

5. COMPLETE MOMENT ACCELERATION

The hyperbolic product and its logarithmic derivative [3, Section 4.36] give

$$\sinh(t/2) = \frac{t}{2} \prod_{\ell \geq 1} \left(1 + \frac{t^2}{4\pi^2 \ell^2}\right), \quad d(t) = 2 \sum_{\ell \geq 1} \frac{(2\pi\ell)^2 - t^2}{((2\pi\ell)^2 + t^2)^2}.$$

Thus a classical infinite product, applied to the regularized original geometric kernel, yields both the positivity bound and a controlled Taylor tail. The product is used to control the Mellin kernel and its remainder.

Write

$$a_r = \frac{(2r+1)B_{2r+2}}{(2r+2)!} = \frac{2(-1)^r(2r+1)\zeta(2r+2)}{(2\pi)^{2r+2}}.$$

Then $d(t) = \sum_{r \geq 0} a_r t^{2r}$ for $|t| < 2\pi$, and

$$\mathcal{R}_d(x) = \sum_{r \geq 0} \frac{a_r x^{2r}}{\zeta(4r+2)} \quad (|x| < 2\pi).$$

For $r \geq 1$, the special values $1/\zeta(4r+2)$ can also be written as $2/(F_2(4r+2) - C_2(4r+2))$, at ordinary convergent arguments. For $r = 0$, evaluate the analytically continued difference at $s = 2$: the individual functions have poles there, while their difference is $2\zeta(2)$. Undefined individual pole values are not subtracted.

Theorem 5.1. *Let $N, K \geq 1$, $x > 0$ and $\tau = x/(N+1)^2 < 2\pi$. Define*

$$R_{N,K}(x) = \sum_{n \leq N} \frac{\mu(n)}{n^2} d(x/n^2) + \sum_{r=0}^{K-1} a_r x^{2r} \left(\frac{1}{\zeta(4r+2)} - \sum_{n \leq N} \frac{\mu(n)}{n^{4r+2}} \right).$$

With $u = (\tau/(2\pi))^2$, put

$$C_K(\tau) = \frac{2\zeta(2K+2)}{(2\pi)^{2K+2}} \left(\frac{2K+1}{1-u} + \frac{2u}{(1-u)^2} \right).$$

The complete infinite transform satisfies

$$(5.1) \quad |\mathcal{R}_d(x) - R_{N,K}(x)| \leq \frac{C_K(\tau)x^{2K}}{(4K+1)N^{4K+1}}.$$

For comparison, the raw cutoff error is at most $1/(12N)$, uniformly in x .

Proof. The product-derived coefficient formula bounds the Taylor remainder at $0 \leq t \leq \tau$ by $C_K(\tau)t^{2K}$, summing $\sum_{r \geq K}(2r+1)u^r$. Sum that remainder over $n > N$ against $|\mu(n)|/n^2$, and use $\sum_{n > N} n^{-4K-2} \leq N^{-4K-1}/(4K+1)$. The lower Taylor terms sum exactly to the displayed reciprocal-zeta corrections. Lemma 3.1 gives the raw bound. $\square$

This is complete-tail acceleration, not an estimate for the large- x arithmetic cancellation. Allowing N to grow with x improves numerical evaluation; an unconditional bound on the resulting finite weighted sum is still needed for Theorem 4.1.

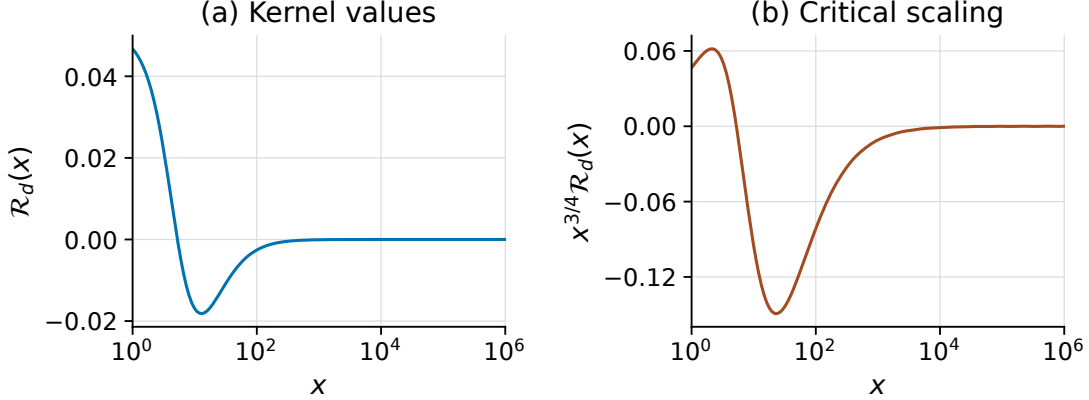


FIGURE 1. Finite evaluations of the complete derivative detector using five exact moment corrections. The right panel uses the critical exponent $3/4$. The computation reaches $x = 10^6$, with explicit omitted-tail estimates; floating arithmetic is not interval-certified. A finite plot does not establish a growth law for all x .

6. HIGHER INTERPOLATION AND RECIPROCAL-ZETA POLES

Let $q, L \geq 1$ be integers. For $n \geq 1$ put $j_n = \lfloor n^{1/q} \rfloor$ and $t_n = n^{1/q} - j_n$. On the real nodes $0, \dots, L-1$, define

$$\ell_r(t) = \prod_{\substack{0 \leq h < L \\ h \neq r}} \frac{t-h}{r-h}, \quad U_{q,L}(s) = \sum_{n \geq 1} \mu(n) \sum_{r=0}^{L-1} \ell_r(t_n)(j_n+r)^{-s}.$$

The $L = 2$ formula is the weighted interpolation $(1-t_n)j_n^{-s} + t_n(j_n+1)^{-s}$. Integer hits have $t_n = 0$, so endpoint handling is exact.

Theorem 6.1. *The series is absolutely convergent on $\operatorname{Re} s > q$. There is a holomorphic function $E_{q,L}$ on $\operatorname{Re} s > q - L$ such that*

$$(6.1) \quad U_{q,L}(s) = \frac{1}{\zeta(s/q)} + E_{q,L}(s)$$

by meromorphic continuation there. Every nontrivial Riemann zero ρ for which $\operatorname{Re}(q\rho) > q-L$ becomes an uncanceled pole at $s = q\rho$, of the same order. Taking $L = q$ includes the full nontrivial-zero strip. At a simple zero the residue is $q/\zeta'(\rho)$.

Proof. Polynomial reproduction of degree less than L is exactly the higher-moment cancellation mechanism. The integral formula for divided differences on real nodes bounds the interpolation remainder of the complex-valued function v^{-s} by

$$\left| \sum_r \ell_r(t)(j+r)^{-s} - (j+t)^{-s} \right| \leq B_L |(s)_L| j^{-\operatorname{Re} s - L},$$

where $B_1 = 1$ and $B_L = 1/(4L)$ for $L \geq 2$. Indeed $t(1-t) \leq 1/4$ and the remaining factors in the node product are at most $2, \dots, L-1$. The integral remainder, rather than a real mean-value assertion for complex values, justifies the modulus bound.

There are $f_q(j) \leq (2^q - 1)j^{q-1}$ inputs in block j . The difference series defining $E_{q,L}$ therefore converges normally on $\operatorname{Re} s > q - L$. On $\operatorname{Re} s > q$, subtract the usual Möbius Dirichlet series and obtain (6.1). Analytic uniqueness continues it. Adding a holomorphic error cannot cancel a pole of the reciprocal zeta function. $\square$

At a completed cutoff through block J , with $\eta = \operatorname{Re} s + L - q > 0$, the omitted error satisfies

$$|E_{q,L}(s) - E_{q,L}^{(J)}(s)| \leq \frac{B_L |(s)_L| (2^q - 1)}{\eta} J^{-\eta}.$$

The finite block coefficients are

$$W_{j,r} = \sum_{j^q \leq n < (j+1)^q} \mu(n) \ell_r(n^{1/q} - j).$$

Expand the Lagrange polynomial and define the finite weighted prefixes by

$$\ell_r(v - j) = \sum_{k=0}^{L-1} a_{j,r,k} v^k, \quad P_k(N) = \sum_{1 \leq n \leq N} \mu(n) n^{k/q}.$$

Direct finite algebra then gives

$$W_{j,r} = \sum_{k=0}^{L-1} a_{j,r,k} (P_k((j+1)^q - 1) - P_k(j^q - 1)).$$

Every sum in this identity is finite, so no rearrangement of an infinite series is involved. The coefficients $a_{j,r,k}$ come from the explicit Lagrange polynomials. Bounds for periodic-character prefixes do not imply bounds for these Möbius-weighted prefixes.

For $q = 2, L = 2$, the error is holomorphic on $\operatorname{Re} s > 0$. All nontrivial Riemann zeros are represented as poles of a floor-ceiling interpolation family, with critical line $\operatorname{Re} s = 1$. RH is equivalent to this family's holomorphy on $\operatorname{Re} s > 1$. This is an exact transport formulation; proving that holomorphy still requires an arithmetic estimate.

7. POLYNOMIALLY SMOOTHED JORDAN-TOTIENT SUMS

The ceiling counts have cumulative sum $\sum_{j=1}^n c_q(j) = n^q$. Apply divisor Möbius inversion to define the classical Jordan totient

$$J_q(n) = \sum_{d|n} \mu(d) (n/d)^q = n^q \prod_{p|n} (1 - p^{-q}) > 0.$$

Its Dirichlet series is $\zeta(s - q)/\zeta(s)$ for $\operatorname{Re} s > q + 1$. The case $q = 1$ is Euler's totient φ [3, Section 27.4]. Every nontrivial zero ρ of the denominator is an uncanceled pole, because $\zeta(\rho - q) \neq 0$.

For integers $k \geq 1, r \geq 0$, define the polynomially smoothed sum

$$T_{q,k,r}(x) = \sum_{n \leq x} J_q(n) (n/x)^r (1 - n/x)^k.$$

Its exact finite reduction is

$$T_{q,k,r}(x) = \sum_{d \leq x} \mu(d) \sum_{h=0}^k (-1)^h \binom{k}{h} (d/x)^{r+h} P_{q+r+h}(\lfloor x/d \rfloor),$$

where $P_m(N) = \sum_{j=1}^N j^m$. The quantities P_m are ordinary finite power sums [1]; no infinite rearrangement is used.

Proposition 7.1. *On $\operatorname{Re} s > q + 1$,*

$$\int_1^\infty T_{q,k,r}(x)x^{-s-1}dx = \frac{k!\zeta(s-q)}{(s+r)(s+r+1)\cdots(s+r+k)\zeta(s)}.$$

Let $C_{q,k,r} = k!(q+r)!/((q+r+k+1)!\zeta(q+1))$. If for every $\epsilon > 0$

$$T_{q,k,r}(x) - C_{q,k,r}x^{q+1} = O_\epsilon(x^{1/2+\epsilon}),$$

then RH follows.

Proof. Integrate each summand after setting $u = n/x$; the beta integral gives the rational kernel. Absolute convergence justifies interchange in the initial half-plane. The assumed error has a Mellin transform holomorphic on $\operatorname{Re} s > 1/2$. Subtracting $C_{q,k,r}/(s-q-1)$ removes the main pole. No rational kernel factor vanishes at a nontrivial Riemann zero, and the numerator $\zeta(\rho-q)$ is nonzero. Thus a zero with real part greater than $1/2$ would give an uncanceled pole of a holomorphic transform. Reflection concludes the argument. $\square$

This is a sufficient criterion. Its required error bound is not proved here. Classical smoothed-totient studies already connect such arithmetic errors with zeta zeros [10]; we use a polynomial kernel and an explicit finite count reduction, rather than importing an RH-conditional estimate as a proof.

7.1. An explicit balanced kernel removes the Mertens cutoff term. For $q = 1, k = 3$, put $R_3 = T_{1,3,0}$ and $B = T_{1,3,1}$. Then $B = T_{1,3,0} - T_{1,4,0}$. Their main terms are $x^2/(20\zeta(2))$ and $x^2/(60\zeta(2))$. For $y > 0$, let $t = \{y\}$ and use standard Bernoulli polynomials. Direct finite power sums give

$$(7.1) \quad H(y) := \sum_{j \leq y} j(1-j/y)^3 = \frac{y^2}{20} - \frac{1}{12} + \frac{1/40 - B_4(t)/4}{y^2} + \frac{B_5(t)}{5y^3},$$

$$(7.2) \quad H_{\text{bal}}(y) := \sum_{j \leq y} \frac{j^2}{y} (1-j/y)^3 = \frac{y^2}{60} + \frac{-1/40 - B_4(t)/4}{y^2} + \frac{2B_5(t)}{5y^3} + \frac{1/252 - B_6(t)/6}{y^4}.$$

The exact divisor formulas are $R_3(x) = \sum_{d \leq x} \mu(d)H(x/d)$ and $B(x) = \sum_{d \leq x} \mu(d)H_{\text{bal}}(x/d)$. Equation (7.1) contains $-M(x)/12$, where $M(x) = \sum_{d \leq x} \mu(d)$. Equation (7.2) has no constant term, hence no such unsmoothed prefix term. The finite difference of smoothing kernels cancels it.

The balanced Mellin multiplier is

$$\frac{6}{(s+1)(s+2)(s+3)(s+4)},$$

so the cancellation has not removed any nontrivial denominator pole. It simplifies the arithmetic expression while preserving detection of every Riemann zero. It does not establish the square-root error bound: weighted Möbius sums and the tail of $\sum \mu(d)/d^2$ remain.

8. A RECIPROCAL-FIBONACCI KERNEL FAMILY

Let $F_1 = F_2 = 1$ and $F_{j+2} = F_{j+1} + F_j$ be the Fibonacci sequence. Define the reciprocal moments $S(m) = \sum_{j \geq 1} j^m/F_j$. The associated geometric and Dirichlet transforms are

$$h_m(t) = \sum_{j \geq 1} \frac{j^m}{F_j} e^{-jt}, \quad Z_{\text{Fib}}(s) = \sum_{j \geq 1} \frac{j^{-s}}{F_j}.$$

The second function is entire, and $h_m(t) = \sum_{k \geq 0} (-t)^k S(m+k)/k!$ for $|t| < \log \varphi$, where $\varphi = (1 + \sqrt{5})/2$. Its Mellin transform is $\Gamma(s)Z_{\text{Fib}}(s-m)$ on $\operatorname{Re} s > 0$.

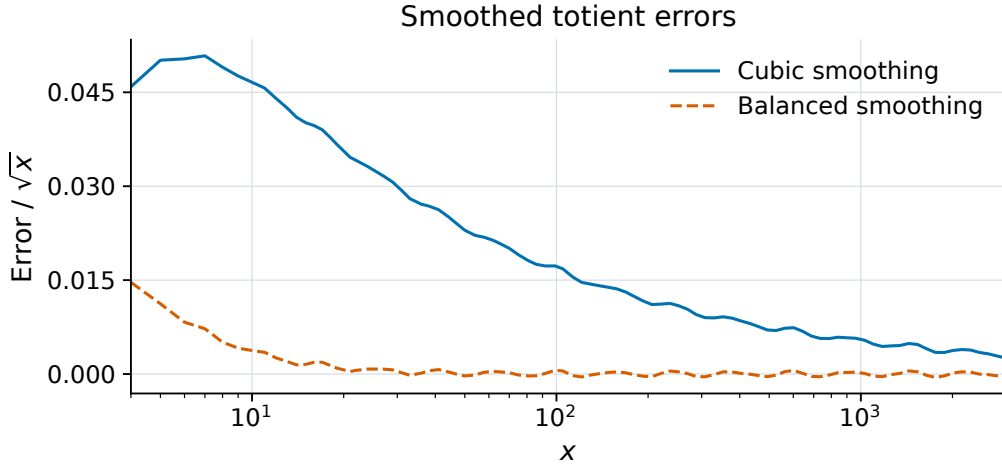


FIGURE 2. Finite cubic and balanced totient errors divided by $\sqrt{x}$. The finite polynomial sums are evaluated with exact integers before numerical main-term subtraction. This illustrates the cancellation construction over a finite range; the required all- x bound remains open.

Theorem 8.1. *For every fixed $s \in \mathbb{C}$, $Z_{\text{Fib}}(s - m) \neq 0$ for all sufficiently large integers m . In particular the family of shifts has no common zero. With $a = \log \varphi$,*

$$Z_{\text{Fib}}(s - m) = \sqrt{5} \frac{\Gamma(m + 1 - s)}{a^{m+1-s}} \left(1 + O_s(m^{-1/2})\right).$$

Proof. Binet's formula gives $F_j^{-1} = \sqrt{5}e^{-aj} + O(e^{-3aj})$ with an absolute constant. Put $u = m - \text{Re } s$ and $\theta = \text{Im } s$. For $f(v) = v^{u-i\theta}e^{-av}$, the difference between its sum over positive integers and its integral is bounded by $\int_0^\infty |f'(v)| dv$. For $u > 0$ this is at most

$$2(u/a)^u e^{-u} + |\theta| \Gamma(u)/a^u.$$

The integral is $\Gamma(m + 1 - s)/a^{m+1-s}$. Stirling's formula makes the relative error $O_s(u^{-1/2})$. The Binet remainder is bounded by a constant times $\sum j^u e^{-3aj}$, whose ratio to the main integral is $O_s(3^{-m})$ by the same sum-integral estimate. The displayed asymptotic and eventual nonvanishing follow. $\square$

Define the ordinary convergent weighted kernels

$$\mathcal{R}_{\text{Fib},m}(x) = \sum_{n \geq 1} \frac{\mu(n)}{n^2} h_m(x/n^2).$$

Their Mellin transforms on $0 < \text{Re } s < 1/2$ are

$$\frac{\Gamma(s) Z_{\text{Fib}}(s - m)}{\zeta(2 - 2s)}.$$

Corollary 8.2. *RH is equivalent to the following family of bounds: for every fixed integer $m \geq 0$ and every $\epsilon > 0$,*

$$\mathcal{R}_{\text{Fib},m}(x) = O_{m,\epsilon}(x^{-3/4+\epsilon}).$$

Proof. Assume the bounds. If RH fails, reflection gives a zero ρ with $\text{Re } \rho > 1/2$. Its mapped point $s_\rho = 1 - \rho/2$ has real part less than $3/4$. Theorem 8.1 supplies a fixed m with $Z_{\text{Fib}}(s_\rho - m) \neq 0$. Thus the continued Mellin quotient has an uncanceled pole at s_ρ . On the other hand, the kernel is bounded at zero and the assumed decay makes its Mellin transform holomorphic on $0 < \text{Re } s < 3/4$, a contradiction.

Conversely assume RH. Mellin inversion and the same contour shift as in Theorem 4.2 give each fixed- m bound. The entire Dirichlet numerator is bounded on every vertical line by its absolutely convergent coefficient sum; Gamma supplies exponential vertical decay. $\square$

Thus reciprocal-Fibonacci moments supply a second RH-equivalent family. A single fixed moment is not asserted to detect every zero. This construction is a systematic application of classical Binet, Mellin and Riesz-type methods [9], with reciprocal-Fibonacci moments as the explicit kernel source.

9. EXACT FLOOR DIFFERENCES AND HILBERT-SPACE APPROXIMATION

9.1. The balanced integer-scale basis. For $n \geq 2$ and $0 < x \leq 1$, define

$$g_n(x) = \frac{1}{n} \left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{nx} \right\rfloor = \left\{ \frac{1}{nx} \right\} - \frac{1}{n} \left\{ \frac{1}{x} \right\}.$$

These are balanced Beurling functions at reciprocal integer scales. Except at endpoints of measure zero, on the block $j = \lfloor 1/x \rfloor$,

$$g_n(x) = \frac{j \bmod n}{n}.$$

Indeed $1/x \in [j, j+1)$ implies $\lfloor 1/(nx) \rfloor = \lfloor j/n \rfloor$. Thus $0 \leq g_n < 1$, and the block length is $1/[j(j+1)]$.

Proposition 9.1. For $\operatorname{Re} s > 0$,

$$(9.1) \quad \int_0^1 g_n(x) x^{s-1} dx = \frac{n^{-1} - n^{-s}}{s} \zeta(s) = \frac{n^{-1} - n^{-s}}{2s} (F_2(s) - C_2(s)).$$

The value at the removable point $s = 1$ is $\log n/n$.

Proof. For $\operatorname{Re} s > 1$ and $0 < a \leq 1$, write $\lfloor a/x \rfloor = \sum_{k \geq 1} \mathbf{1}_{x \leq a/k}$ and integrate to obtain $\int_0^1 \lfloor a/x \rfloor x^{s-1} dx = a^s \zeta(s)/s$. Subtraction proves (9.1) initially. Boundedness of g_n makes the left side holomorphic on $\operatorname{Re} s > 0$; the numerator removes the pole at 1 on the right. Analytic uniqueness and (2.7) prove the continued identity. Expansion at 1 gives the stated mean. $\square$

9.2. The obstruction from an off-line zero. For finite coefficients put

$$p_N(x) = \sum_{n=2}^N c_n g_n(x), \quad \epsilon_N^2 = \int_0^1 |1 - p_N(x)|^2 dx.$$

Theorem 9.2. If $\zeta(\rho) = 0$ and $\beta = \operatorname{Re} \rho > 1/2$, then

$$(9.2) \quad \epsilon_N^2 \geq \frac{2\beta - 1}{|\rho|^2}.$$

For any nontrivial zero $\rho = \beta + it$, reflection gives

$$(9.3) \quad |\beta - 1/2| \leq \frac{1 + t^2}{2} \epsilon_N^2.$$

An unconditional construction with $\epsilon_N \rightarrow 0$ would prove RH.

Proof. Equation (9.1) gives $\int_0^1 p_N(x) x^{\rho-1} dx = 0$, hence $\int_0^1 (1 - p_N(x)) x^{\rho-1} dx = 1/\rho$. Cauchy-Schwarz and $\int_0^1 |x^{\rho-1}|^2 dx = (2\beta - 1)^{-1}$ give (9.2). For $\beta > 1/2$ use $|\rho|^2 \leq 1 + t^2$; for $\beta < 1/2$, apply the same argument to $1 - \bar{\rho}$. The case $\beta = 1/2$ is immediate. This proves (9.3). A vanishing error excludes zeros to the right of the critical line, and reflection excludes those to the left. $\square$

The finite-height inequality is quantitative, but may be weaker than known location results for the computed errors. Only an infinite convergence estimate gives the critical line for all heights.

9.3. The classical approximation criterion. Let d_N be the minimum of ϵ_N over the span of $g_2, \dots, g_N$.

Theorem 9.3 (Nyman–Beurling–Báez-Duarte).

$$\text{RH} \iff d_N \longrightarrow 0.$$

Proof. The integer-scale criterion in [6, Theorem 1.1] uses $\rho_n(x) = \{1/(nx)\}$ and the target $\chi = \mathbf{1}_{(0,1)}$ in $L^2(0, \infty)$. Extend g_n by $\rho_n - \rho_1/n$; it vanishes on $x > 1$ and belongs to that integer-scale span. Conversely an approximant $f = \sum a_n \rho_n$ has tail A/x on $x > 1$, where $A = \sum a_n/n$. Thus $|A| \leq \|f - \chi\|_2$. Replacing f by $f - A\rho_1$ gives a balanced approximant and changes the norm by at most $|A|\|\rho_1\|_2$. Approximation in the integer-scale span is therefore equivalent to approximation in the balanced span. Apply the cited theorem. $\square$

9.4. The exact Gram matrix. Let G_N be the Gram matrix indexed by $2, \dots, N$ and let $b_n = \langle 1, g_n \rangle = \log n/n$.

Proposition 9.4. For $L = \text{lcm}(m, n)$,

$$(9.4) \quad G_{mn} = \sum_{j \geq 1} \frac{(j \bmod m)(j \bmod n)}{mn} \left(\frac{1}{j} - \frac{1}{j+1} \right),$$

$$(9.5) \quad = \frac{1}{mnL} \sum_{r=1}^L (r \bmod m)(r \bmod n) \left[\psi\left(\frac{r+1}{L}\right) - \psi\left(\frac{r}{L}\right) \right],$$

where $\psi = \Gamma'/\Gamma$. The matrix G_N is positive definite. If K_N is the extended Gram matrix of $1, g_2, \dots, g_N$, then

$$(9.6) \quad d_N^2 = 1 - b^T G_N^{-1} b = \frac{\det K_N}{\det G_N}.$$

Proof. Block integration gives (9.4). Its coefficient is periodic with period L . Split into residues and sum $1/(kL + r) - 1/(kL + r + 1)$ for $k \geq 0$. The convergent difference series for the digamma function gives (9.5); the term $r = L$ has zero coefficient.

If $\sum c_n g_n = 0$, Mellin transformation on $\text{Re } s > 1$ gives $\sum c_n (n^{-1} - n^{-s}) = 0$. Let real $s \rightarrow \infty$ to remove the constant, then successively isolate the smallest base in the finite Dirichlet polynomial. Every coefficient is zero. Linear independence proves positive definiteness. Completing the square gives the first expression in (9.6); the Schur complement gives the determinant ratio. $\square$

These formulas give an explicit realization of the classical Gram problem; see [7] for related decompositions. Finite positivity does not prove that the determinant ratio vanishes.

9.5. An arithmetic obstruction to dyadic scales.

Proposition 9.5. Every finite approximant using only g_{2^k} , $k \geq 1$, satisfies

$$\|1 - p\|_2^2 \geq \frac{1}{1 + (2\pi/\log 2)^2} = 0.01202368851982670579 \dots$$

Thus 1 is not in the closure of the dyadic-scale span.

Proof. At $s_m = 1 + 2\pi im/\log 2$, $m \neq 0$, the factor $2^{-k} - 2^{-ks_m}$ in (9.1) vanishes for every k . The zeta factor is finite. The same Cauchy–Schwarz calculation gives the bound with $|s_m|^{-2}$; take $m = 1$. $\square$

The obstruction is an artificial common zero of the basis transforms. It holds independently of RH.

9.6. A finite Möbius candidate. Take $c_n = -\mu(n)$ for $2 \leq n \leq N$ and put $A_N = \sum_{n \leq N} \mu(n)/n$.

Proposition 9.6. *For $1 \leq j \leq N$,*

$$1 - \sum_{n=2}^N (-\mu(n)) \frac{j \bmod n}{n} = jA_N.$$

The contribution of these blocks to the squared error is

$$A_N^2(N+1 - H_{N+1}).$$

Proof. Write $j \bmod n = j - n \lfloor j/n \rfloor$ and use $\sum_{n \leq j} \mu(n) \lfloor j/n \rfloor = 1$. The block contribution is $A_N^2 \sum_{j=1}^N j/(j+1) = A_N^2(N+1 - H_{N+1})$. $\square$

This identity controls only the displayed blocks. Pointwise convergence does not supply a uniform estimate for the remaining norm. The natural unsmoothed Möbius approximation fails in the full Hilbert-space problem [6]. Sharp estimates for some smoothed approximants use RH and additional derivative assumptions [8]; those conditional estimates are not used as unconditional bounds here.

9.7. Orthogonal updates. Let P_N be orthogonal projection onto the current span, let $r_N = 1 - P_N 1$, and put $v_{N+1} = g_{N+1} - P_N g_{N+1}$.

Proposition 9.7.

$$d_{N+1}^2 = d_N^2 - \frac{|\langle r_N, v_{N+1} \rangle|^2}{\|v_{N+1}\|^2}.$$

Proof. The next span is the orthogonal sum of the current span and the nonzero vector v_{N+1} . Project the residual onto that vector and use Pythagoras. $\square$

The subtracted terms are nonnegative. Proving that their total exhausts the initial squared norm is equivalent to proving $d_N \rightarrow 0$. Monotonicity alone does not determine the limit.

10. REAL-ZERO ASYMPTOTICS AND COMPLEX-ZERO MOTION

10.1. Far-left real zeros. The functional equation explains the negative-real pattern much more precisely. The relevant anchor is the highest shift $q\zeta(s-d)$, whose trivial zeros are $d-2n$.

Theorem 10.1. *Fix $q \geq 2$ and $d = q-1$. For every sufficiently large integer n , each of F_q and C_q has exactly one zero in $|s - (d-2n)| < 1/2$. These zeros are simple and real. Writing them as $s_{q,n}^F$ and $s_{q,n}^C$,*

$$(10.1) \quad s_{q,n}^F = d - 2n - \frac{d}{n} + O_q(n^{-3}),$$

$$(10.2) \quad s_{q,n}^C = d - 2n + \frac{d}{n} + O_q(n^{-3}).$$

More precisely, with $\sigma = -1$ for F and $\sigma = +1$ for C ,

$$(10.3) \quad s_{q,n}^\sigma = d - 2n + \sigma \frac{d}{n} + \frac{d^2/2 + \sigma \pi^2 d(d-2)/12}{n^3} + O_q(n^{-4}).$$

The constants depend on fixed q ; this is not an estimate uniform in growing q .

Proof. Use

$$\zeta(w) = 2^w \pi^{w-1} \sin(\pi w/2) \Gamma(1-w) \zeta(1-w)$$

from [3, Section 25.4]. Divide the sum by the nonvanishing factor $q2^{s-d} \pi^{s-d-1} \Gamma(1-s+d)$ near $s = d-2n$. Put $N = 2n$, $s = d - N + \delta$, $\theta = \pi\delta/2$, and

$$b_r = \frac{1}{q} \binom{q}{d-r} \quad (0 \leq r \leq d), \quad P_r(\delta) = \prod_{h=0}^{r-1} (N - h - \delta), \quad P_0 = 1.$$

The resulting function, after removing the sign $(-1)^n$, is exactly

$$(10.4) \quad S_N(\delta) \sin \theta + \varepsilon T_N(\delta) \cos \theta,$$

where $\varepsilon = +1$ for F and $\varepsilon = -1$ for C , and

$$S_N(\delta) = \sum_{\substack{0 \leq r \leq d \\ r \text{ even}}} \frac{(-1)^{r/2} b_r (2\pi)^r}{P_r(\delta)} \zeta(N+1-r-\delta),$$

$$T_N(\delta) = \sum_{\substack{0 \leq r \leq d \\ r \text{ odd}}} \frac{(-1)^{(r-1)/2} b_r (2\pi)^r}{P_r(\delta)} \zeta(N+1-r-\delta).$$

For $|\delta| \leq 1/2$, fixed q and large N , the positive-half-plane Dirichlet series gives $\zeta(N+1-r-\delta) = 1 + O_q(2^{-N})$, uniformly. Thus $S_N = 1 + O_q(N^{-2})$ and $T_N = O_q(N^{-1})$. On the circle $|\delta| = 1/2$, the perturbation of $\sin \theta$ is uniformly smaller than $|\sin \theta|$ for large N . Rouché's theorem gives one zero counted with multiplicity. Conjugation symmetry makes that unique zero real, and the count makes it simple.

The same equation first gives $\delta = O_q(N^{-1})$. Since

$$b_1 = \frac{d}{2}, \quad b_2 = \frac{d(d-1)}{6}, \quad b_3 = \frac{d(d-1)(d-2)}{24},$$

with missing terms interpreted as zero, expansion at that root yields

$$S_N = 1 - \frac{(2\pi)^2 b_2}{(N-\delta)(N-1-\delta)} + O_q(N^{-4}) + O_q(2^{-N}),$$

$$T_N = \frac{2\pi b_1}{N-\delta} - \frac{(2\pi)^3 b_3}{(N-\delta)(N-1-\delta)(N-2-\delta)} + O_q(N^{-5}) + O_q(2^{-N}).$$

Solving $\tan(\pi\delta/2) = -\varepsilon T_N/S_N$ through order N^{-3} gives

$$\delta = -\varepsilon \frac{2d}{N} + \frac{4d^2 - \varepsilon(2\pi^2/3)d(d-2)}{N^3} + O_q(N^{-4}).$$

There is no N^{-2} term. Substituting $N = 2n$ and $\sigma = -\varepsilon$ proves (10.3), hence (10.1)–(10.2). $\square$

Corollary 10.2. *The two far-left zeros satisfy*

$$s_{q,n}^C - s_{q,n}^F = \frac{2d}{n} + \frac{\pi^2 d(d-2)}{6n^3} + O_q(n^{-4}),$$

and their midpoint is $d - 2n + d^2/(2n^3) + O_q(n^{-4})$.

The bend is therefore an explicit reciprocal displacement, with nearly common centers. It is not motion along the exact negative even zero set of $\zeta(s)$. The theorem guarantees a family far to the left, and does not count every other zero of either function.

10.2. A count-polynomial deformation. For fixed integer q , define

$$(10.5) \quad Z_q(u; s) = \sum_{j \geq 1} ((j+u)^q - (j+u-1)^q) j^{-s} = \sum_{k=0}^{q-1} a_k(u) \zeta(s-k),$$

initially for $\operatorname{Re} s > q$, where

$$a_k(u) = \binom{q}{k} (u^{q-k} - (u-1)^{q-k}).$$

Then $Z_q(0; s) = C_q(s)$ and $Z_q(1; s) = F_q(s)$. This is a deformation of the multiplicity polynomial, not a claim that intermediate u produces actual floor or ceiling counts. At $u = 1/2$, only terms for which $q-k$ is odd remain. At sufficiently negative integers, all those retained shifts have the same zeta-zero parity: the midpoint family has zeros at every negative even integer for odd q , and at every negative odd integer for even q .

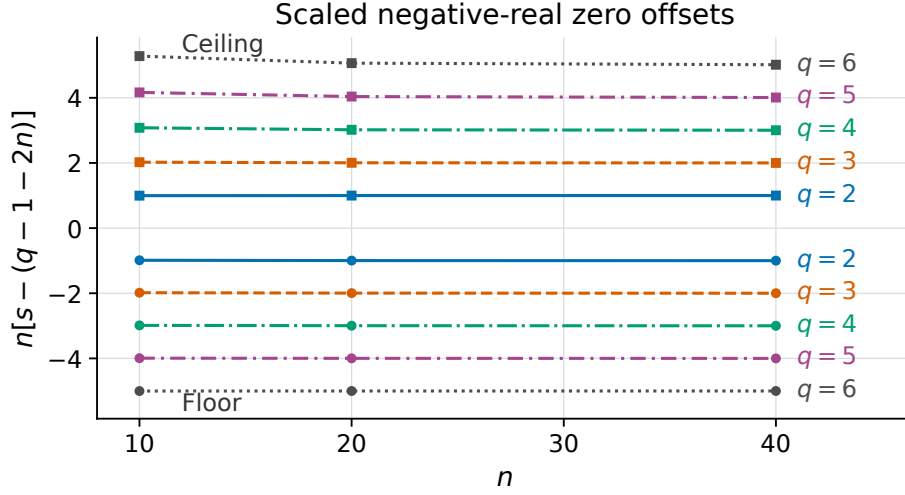


FIGURE 3. Computed scaled offsets $n(s - (q - 1 - 2n))$ for $q = 2, \dots, 6$ and $n = 10, 20, 40$. They approach $-(q - 1)$ for the floor function and $+(q - 1)$ for the ceiling function. The formulas are proved in Theorem 10.1; the plotted points are numerical checks.

Proposition 10.3. *Let $Z(u; s)$ be holomorphic near (u_0, r_0) , with $Z(u_0; r_0) = 0$ and $Z_s(u_0; r_0) \neq 0$. There is a local analytic zero branch $r(u)$, and*

$$(10.6) \quad r'(u) = -\frac{Z_u}{Z_s},$$

$$(10.7) \quad r''(u) = -\frac{Z_{uu} + 2Z_{su}r' + Z_{ss}(r')^2}{Z_s}.$$

All quantities on the right are evaluated at $(u, r(u))$. If $r' \neq 0$, the geometric curvature for real u is

$$\kappa(u) = \frac{|\operatorname{Im}(r''\overline{r'})|}{|r'|^3}.$$

Proof. The holomorphic implicit function theorem supplies the branch. Differentiate the identity $Z(u; r(u)) = 0$ once and twice. Apply the usual plane curve curvature formula to the real and imaginary parts. $\square$

This method of studying zeros through parameter derivatives is classical; Frisk and de Gosson [4] apply it to Hurwitz zeta functions and their combinations. Here the count polynomials give the specific derivative coefficients.

10.3. Anchoring a branch at a shifted Riemann zero. For $X = F, C$, let $R_{q,X}(s) = X_q(s) - q\zeta(s - d)$ and introduce

$$(10.8) \quad W_{q,X}(\lambda; s) = q\zeta(s - d) + \lambda R_{q,X}(s).$$

At $\lambda = 0$ its zeros are exactly the shifted Riemann zeros; at $\lambda = 1$ it is the rounded function. If ρ is a simple Riemann zero, the branch from $r_0 = d + \rho$ has

$$(10.9) \quad r'(0) = -\frac{R_{q,X}(d + \rho)}{q\zeta'(\rho)}.$$

A sufficient local existence test throughout $|\lambda| \leq 1$ is

$$|R_{q,X}(s)| < q|\zeta(s - d)| \quad (s \in \partial D),$$

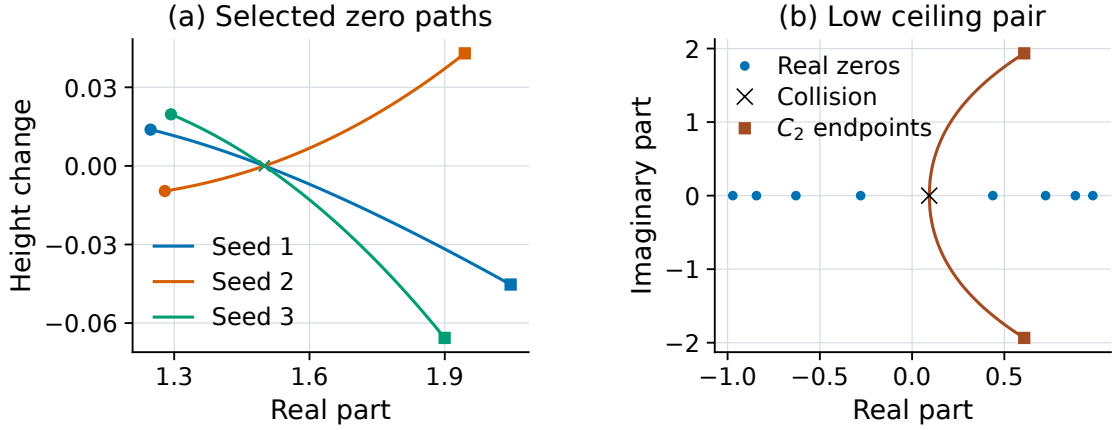


FIGURE 4. Left: computed paths of the first three shifted nontrivial Riemann zeros under (10.10), with their starting heights subtracted. Squares mark the ceiling endpoint, crosses the midpoint and circles the floor endpoint. Right: the low conjugate pair discussed in Section 10.4, with sampled real precursor branches and their collision. These are selected numerical paths, not a complete zero census.

where the closed disk D contains one simple shifted zero and no pole. Rouché's theorem then preserves one zero in the disk. If this inequality is not proved, numerical continuation provides evidence but not a certified branch count.

For $q = 2$ the particularly transparent signed family is

$$(10.10) \quad G(\lambda; s) = 2\zeta(s-1) + \lambda\zeta(s), \quad -1 \leq \lambda \leq 1.$$

Its ceiling, midpoint and floor endpoints are C_2 , $2\zeta(s-1)$ and F_2 . From a simple zero $1 + \rho$,

$$(10.11) \quad r'(0) = -\frac{\zeta(1+\rho)}{2\zeta'(\rho)},$$

$$(10.12) \quad r''(0) = -\frac{\zeta''(\rho)(r'(0))^2 + \zeta'(1+\rho)r'(0)}{\zeta'(\rho)}.$$

The floor and ceiling displacements are opposite to first order, but the quadratic term is the same. This is an exact mechanism for curved, asymmetric trajectories. Neither the direction of bending nor a height-asymptotic displacement follows from q alone: the zeta values and derivatives depend on the individual seed.

10.4. The low ceiling pair and a multiple-zero event. The pair printed for C_2 in [2] is reproduced as

$$(10.13) \quad s = 0.607804816070581 \pm 1.935001090218557i.$$

Its height is far below the first nontrivial Riemann-zero height. Within (10.10), however, it has a natural explanation.

For real $-1 < s < 1$, set

$$(10.14) \quad L(s) = -\frac{2\zeta(s-1)}{\zeta(s)}.$$

The denominator is negative and nonzero on this interval. The numerator's zeta factor is also negative. The continuous extension satisfies $L(-1) = L(1) = 0$ and $L(0) = -1/3$. The endpoint at 1 is a limit, since ζ has a pole there. Thus L has a negative interior minimum. At every interior stationary point s_c , putting $\lambda_c = L(s_c)$ gives

$$G(\lambda_c; s_c) = G_s(\lambda_c; s_c) = 0.$$

This proves the existence of at least one real multiple-zero event in the deformation. The sign statements follow from the zeta functional equation on $(-2, 0)$ and the alternating eta series on $(0, 1)$.

At small negative λ , two local real branches have different origins:

$$(10.15) \quad r_{\text{pole}}(\lambda) = 1 + \lambda + O(\lambda^2),$$

$$(10.16) \quad r_{\text{triv}}(\lambda) = -1 - \lambda \frac{\zeta(-1)}{2\zeta'(-2)} + O(\lambda^2).$$

For (10.15), apply the implicit theorem to $(s-1)G$ near $(0, 1)$. At $\lambda = 0$ its zero at 1 is caused by removing a pole factor; $G(0; 1) = -1$ is not zero. For (10.16), apply the theorem directly at the simple shifted trivial zero $s = -1$. The first branch tends to a disappearing pole location; the second tends to a shifted trivial zero.

Proposition 10.4. *If $G = G_s = 0$ at a real (λ_c, s_c) and $G_{ss}G_\lambda \neq 0$ there, then the two local branches have*

$$(s - s_c)^2 = K(\lambda - \lambda_c) + O(|\lambda - \lambda_c|^{3/2}), \quad K = -\frac{2G_\lambda}{G_{ss}}.$$

When $K > 0$, the nearby branches are real on the $\lambda > \lambda_c$ side and form a conjugate pair on the $\lambda < \lambda_c$ side.

Proof. Taylor expansion has the leading terms $G_\lambda(\lambda - \lambda_c) + G_{ss}(s - s_c)^2/2$. The analytic local factorization at a nondegenerate double zero gives two square-root branches with the stated remainder. Real coefficients and the sign of the leading squared displacement give the real versus conjugate distinction. $\square$

The high-precision calculation finds

$$\begin{aligned} s_c &\simeq 0.09275171261477142275, \\ \lambda_c &\simeq -0.33562691098557512999, \\ G_{ss} &\simeq 0.32283371219899411870, \quad G_\lambda \simeq -0.59474347018104348662, \\ K &\simeq 3.68451898118028443613. \end{aligned}$$

Numerical continuation from the double-zero neighborhood reaches (10.13) at $\lambda = -1$, while the two real branches head toward -1 and 1 as $\lambda \uparrow 0$. The critical coordinates and the full connecting paths are numerical findings, not interval-certified uniqueness or continuation theorems. The existence of a multiple-zero event and the local mechanism under the explicit nondegeneracy hypotheses are analytic statements.

10.5. Selected complex roots. The supplied script starts from $q-1+\rho_n$ and follows (10.8) in ten steps to $\lambda = 1$, for $q = 2, \dots, 6$, $n = 1, \dots, 6$ and both rounding choices. Selected endpoint roots are refined with 75 decimal digits of working precision. The recorded residual is

$$\frac{|X_q(s)|}{1 + \sum_{k=0}^{q-1} |a_k \zeta(s-k)|},$$

so cancellation between large summands is measured relative to their scale. Residuals are checked below 10^{-60} . This does not certify the final digits or prove that no branch was missed between steps.

The low ceiling zeros for $q = 2, 3, 4$ are refined separately from the published seeds. Sampled winding-number checks on upper-half-plane boxes containing these low roots are performed at two grid resolutions. Agreement supports the reported counts in those boxes but is not an interval-certified argument-principle calculation. In particular, the observation of $q-1$ low pairs in the original tables does not by itself prove that all extra complex zeros have been found.

TABLE 1. Selected complex roots from the first Riemann seed. Values shown to twelve decimal places; the data file contains the computed digits and scaled residuals.

q	Floor root	Ceiling root
2	$1.247595281028 + 14.148570425918i$	$2.046038604155 + 14.089440977961i$
3	$1.964049664860 + 14.165353520342i$	$3.268324757971 + 14.085906923157i$
4	$2.653604262294 + 14.184700388061i$	$4.376051387994 + 14.092471562169i$
5	$3.321005478954 + 14.206156570648i$	$5.432323332837 + 14.101111461612i$
6	$3.970873284253 + 14.229263328907i$	$6.462895459971 + 14.109191378598i$

TABLE 2. Negative-real checks at $n = 40$. The scaled offset $n(s - (q - 1 - 2n))$ approaches the signed limit in the last column.

q	Function	Scaled offset	Limit
2	F_q	-0.999174769	-1
2	C_q	0.999798487	1
3	F_q	-1.998801665	-2
3	C_q	2.001301982	2
4	F_q	-2.998840708	-3
4	C_q	3.004478349	3
5	F_q	-3.999253072	-4
5	C_q	4.009296172	4
6	F_q	-5.000001020	-5
6	C_q	5.015724711	5

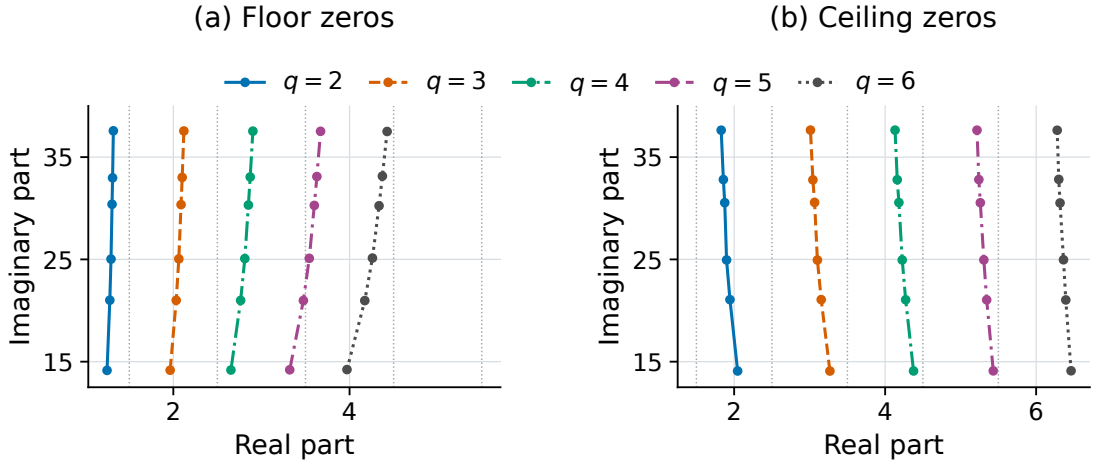


FIGURE 5. Selected endpoint zeros reached from the first six shifted Riemann seeds. Dotted vertical lines are $\text{Re } s = q - 1/2$, the shifted critical-line positions of the starting seeds. Connecting the discrete points highlights the observed bend across height; it does not represent zeros at every intervening height.

11. LITERAL ROUNDING AND PARAMETER MODELS

11.1. **Varying a count polynomial is not varying an actual rounded exponent.** For noninteger positive q , the actual inverse counts at $b = 1$ are

$$\lceil (j+1)^q \rceil - \lceil j^q \rceil, \quad \lfloor j^q \rfloor - \lfloor (j-1)^q \rfloor.$$

These depend on threshold crossings. They are not generally the smooth counts $(j+1)^q - j^q$ and $j^q - (j-1)^q$. Consequently a smooth curve drawn through integer- q roots requires a stated interpolation; it is not automatically a zero trajectory of the literal rounded family.

One useful smooth model is defined by the latter two counts. Removing the $j = 1$ term before expanding gives the rapidly convergent meromorphic representations

$$(11.1) \quad \tilde{F}(q, s) = 2^q - 1 + \sum_{r \geq 1} \binom{q}{r} (\zeta(s - q + r) - 1),$$

$$(11.2) \quad \tilde{C}(q, s) = 1 + \sum_{r \geq 1} (-1)^{r+1} \binom{q}{r} (\zeta(s - q + r) - 1).$$

They initially follow in $\operatorname{Re} s > q$ by expanding at $j \geq 2$. On compact sets away from their polar sets, the tails converge normally: the generalized binomial coefficients have at most polynomial growth while $\zeta(s - q + r) - 1 = O(2^{-r})$. At integer q the sums truncate and recover F_q, C_q . These formulas provide a specified interpolation for investigating derivatives with respect to q .

The discrepancy between smooth and literal counts has bounded cumulative sums. For the floor counts put $e_j = \lceil j^q \rceil - j^q$; their discrepancy is $e_{j+1} - e_j$. For the ceiling counts put $\eta_j = \lfloor j^q \rfloor - j^q$; the discrepancy is $\eta_j - \eta_{j-1}$. Each cumulative sum telescopes to a bounded difference. Abel summation therefore gives a normally convergent discrepancy transform on $\operatorname{Re} s > 0$. Extending either individual literal transform farther left requires additional arithmetic information. The smooth interpolation does not, by itself, define a continuation of the literal rounded function.

11.2. Rational scaling and periodic twists. For integer q and positive rational inverse scale α , the positive-level counts are

$$\lceil \alpha^q (j+1)^q \rceil - \lceil \alpha^q j^q \rceil, \quad \lfloor \alpha^q j^q \rfloor - \lfloor \alpha^q (j-1)^q \rfloor.$$

The rounding discrepancies of these rational polynomials are periodic in j . The counts are therefore polynomial plus periodic. For a common period M , splitting into residues gives finite combinations of shifted Hurwitz zeta functions, since, initially on $\operatorname{Re} s > k + 1$,

$$\sum_{\substack{j \geq 1 \\ j \equiv r \pmod{M}}} j^{k-s} = M^{k-s} \zeta(s - k, r/M).$$

The local derivative formulas apply to holomorphic coefficient deformations of these expressions. Their zero-distribution questions belong to the Hurwitz-combination setting [5]. Distinct rational scales form discrete arithmetic models.

Mean-zero periodic inputs furnish exact additional zero patterns. If h is mean-zero periodic and p is a prime, define a dilation difference by

$$(\Delta_p h)(n) = h(n) - p \mathbf{1}_{p|n} h(n/p).$$

For its periodic Dirichlet transform $L_h(s)$,

$$(11.3) \quad L_{\Delta_p h}(s) = (1 - p^{1-s})^r L_h(s).$$

Mean-zero periodicity gives entire continuation: $L_h(s) = M^{-s} \sum_{r=1}^M h(r) \zeta(s, r/M)$, and the residue at 1 is $M^{-1} \sum h(r) = 0$ [3, Section 25.11]. The initial dilation identity follows by changing variables on $\operatorname{Re} s > 1$. Consequently zeros occur at

$$s = 1 + \frac{2\pi i m}{\log p} \quad (m \in \mathbb{Z}),$$

with order at least r , and exactly r where L_h is nonzero. These are deliberately factored arithmetic zero ladders, distinct from cancellation zeros of the unweighted F_q, C_q . An identically zero transform has no isolated zero set and is excluded from this discussion.

11.3. The spectral variable and the exponent variable. A canonical product built from factors $(1 + z/j)$ with multiplicity $c_q(j)$, after the usual Taylor subtractions, has zeros at negative spectral parameters $z = -j$. Those are not the zeros of $C_q(s)$ in the exponent variable. Their logarithmic expansions contain zeta values and encode the spectrum, but do not immediately locate zeta-function zeros. To build a product over the s -zeros instead, first remove the poles:

$$E_{q,X}(s) = \prod_{j=1}^q (s - j) X_q(s).$$

This is entire and is nonzero at those pole locations because every residue is nonzero. A canonical factorization of this entire function would use its own s -zero set and growth estimates. It is a different product construction. In a deformation, a disappearing residue can introduce an artificial zero of the pole-cleared function, as at $(\lambda, s) = (0, 1)$ in (10.10); that event must be distinguished from a genuine zero of the limiting meromorphic function.

12. NUMERICAL EVIDENCE AND REMAINING ESTIMATES

12.1. Verification conventions. The supplementary programs distinguish exact finite identities, high-precision numerical comparisons, and analytic truncation bounds. Integer or rational checks verify the cutoff identities directly. The zero and kernel evaluations use floating arithmetic; they are not interval-certified global zero counts. The analytic statements follow from the proofs above.

The Gram computations use 65 decimal digits, with systems through $N = 8$ repeated at 85 digits. Direct partial block integrals are checked using explicit tail bounds, together with (9.1), the exact $N = 2$ identity, and the finite Möbius calculation. For $N = 2$, $G_{22} = \log 2/4$,

TABLE 3. Finite balanced approximation errors.

N	Computed d_N^2
2	0.306852819440054691
4	0.065951694215569654
8	0.024244525306522673
12	0.020498147332894166
16	0.017936267020202550
20	0.016538252109392516
24	0.015577777992360689

$b_2 = \log 2/2$, the optimal coefficient is 2, and $d_2^2 = 1 - \log 2$ exactly. Decrease at finite N is consistent with the nested spans and does not determine the limiting error.

Independent kernel checks comprise three regularized Mellin evaluations, 194 exact fractional-cutoff power-sum identities, five pairs of exact totient regroupings, and 1000 higher-interpolation inputs with polynomial reproduction and remainder bounds. Complete-kernel evaluations at two cutoffs agree within the analytic tail bounds. Reciprocal-Fibonacci checks compare the moment generator and the large-moment asymptotic.

12.2. The outstanding arithmetic estimates. The constructions provide exact implications for all nontrivial Riemann zeros. They do not supply the unconditional arithmetic estimates needed to determine the global location parameter. For $\mathcal{R}_d$, the proved baseline is $O(x^{-1/2})$, and the critical target is $O_\epsilon(x^{-3/4+\epsilon})$. For the balanced Beurling basis, the target is a vanishing upper bound for (9.6). For the balanced totient sum, it is a square-root main-term-subtracted error in Proposition 7.1.

The Möbius prefixes in these targets are not periodic character prefixes. Kernel positivity, exact block identities, finite Gram positivity and accurate finite evaluations do not provide the missing uniform cancellation. The far-left zero asymptotic and the local deformation

TABLE 4. Complete-kernel evaluations, with $K = 5$. The last column bounds the analytic omitted tail; it does not enclose floating-point roundoff.

x	N	$R_{N,5}(x)$	Tail bound
1	64	$4.67245312158 \times 10^{-2}$	3.2537×10^{-48}
10	64	$-1.69635250936 \times 10^{-2}$	3.2537×10^{-38}
100	64	$-2.5761101724 \times 10^{-3}$	3.2538×10^{-28}
1000	64	$-6.1244832075 \times 10^{-5}$	3.2592×10^{-18}
10000	200	$-1.06255528095 \times 10^{-6}$	1.3223×10^{-18}
1000000	2000	$6.00764280301 \times 10^{-10}$	1.3224×10^{-19}

TABLE 5. Selected independent checks; all passed.

Check	Count or largest residual
Regularized Mellin integral	3 cases; $< 10^{-64}$
Exact fractional-cutoff identities	194
Exact totient regroupings	5 pairs
Higher interpolation and power reproduction	1000 inputs; $< 10^{-63}$
Two-cutoff complete-tail agreement	6 scales through 10^6
Fibonacci moment-generator check	2.13×10^{-50}

formulas explain specified rounded-zeta zero families; a continuation experiment starting from known critical-line seeds does not certify that every original Riemann zero has been represented.

12.3. Reproducibility. The supplementary material contains the three verification programs, their recorded outputs, the root data and instructions. The source manuscript compiles with standard `pdflatex` without external documents or shell escape. The numerical programs require Python, `mpmath`, `numpy` and `matplotlib`. Explicit analytic tail bounds are stated independently of floating-point roundoff.

REFERENCES

- [1] D. Shah, M. Sahni, R. Sahni, E. León-Castro and M. Olazabal-Lugo, *Series of Floor and Ceiling Function—Part I: Partial Summations*, Mathematics **10** (2022), 1178. <https://doi.org/10.3390/math10071178>.
- [2] D. Shah, M. Sahni, R. Sahni, E. León-Castro and M. Olazabal-Lugo, *Series of Floor and Ceiling Functions—Part II: Infinite Series*, Mathematics **10** (2022), 1566. <https://doi.org/10.3390/math10091566>.
- [3] NIST Digital Library of Mathematical Functions, Sections 4.36, 24.4, 25.4–25.6, 25.10–25.11 and 27.4. <https://dlmf.nist.gov/>.
- [4] H. Frisk and S. de Gosson, *On the Motion of Zeros of Zeta Functions*, arXiv:math-ph/0102007 (2001). <https://arxiv.org/abs/math-ph/0102007>.
- [5] R. Steuding and J. Steuding, *Linear combinations of Hurwitz zeta-functions*, Kyushu J. Math. **76** (2022), 27–41. <https://doi.org/10.2206/kyushujm.76.27>.
- [6] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann Hypothesis*, arXiv:math/0202141v2 (2002). <https://arxiv.org/html/math/0202141v2>.
- [7] W. Ehm, *On certain Gram matrices and their associated series*, arXiv:2405.06349v2 (2024). <https://arxiv.org/abs/2405.06349>.
- [8] S. Bettin, J. B. Conrey and D. W. Farmer, *An optimal choice of Dirichlet polynomials for the Nyman–Beurling criterion*, Proc. Steklov Inst. Math. **280** (2013), suppl. 2, S30–S36. <https://doi.org/10.1134/S0081543813030036>; arXiv:1211.5191. <https://arxiv.org/abs/1211.5191>.
- [9] A. Agarwal, M. Garg and B. Maji, *Riesz-type criteria for the Riemann hypothesis*, Proc. Amer. Math. Soc. **150** (2022), 5151–5163. <https://doi.org/10.1090/proc/16064>; arXiv:2202.00637. <https://arxiv.org/abs/2202.00637>.
- [10] S. Das, H. Lang, H. Wan and N. Xu, *The Distribution of Error Terms of Smoothed Summatory Totient Functions*, arXiv:2207.07722 (2022). <https://arxiv.org/abs/2207.07722>.

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